Dear Colleagues:

On behalf of the Massachusetts Department of Youth Services, and in partnership with the Collaborative for Educational Services and Commonwealth Corporation, I am pleased to provide you with the 2017 edition of the DYS Mathematics Instructional Guide. This Guide is aligned with the 2011 Massachusetts Mathematics Framework and the Common Core State Standards (CCSS), and reflects the recently adopted revisions to the mathematics standards approved by the Massachusetts Department of Elementary and Secondary Education in the spring of 2017.

With this Guide, our goal is to provide DYS educators with a cutting-edge resource that informs planning and instruction of curricula and authentic assessment of student learning. The 2017 DYS Mathematics Instructional Guide provides you with an overview of the 2017 Massachusetts Curriculum Framework for Mathematics aligned with the Common Core State Standards, and guidance for implementing those standards in DYS schools. The Guide features standards-aligned scope and sequences for Algebra 1, Geometry, and Algebra 2, and curriculum unit exemplars adapted for both long- and short-term program settings. Finally, the Mathematics Guide incorporates research-based instructional models that serve as the foundation for our work with DYS youth: Universal Design for Learning, Understanding by Design, Empower Your Future and the DYS Future Ready Framework, Culturally Responsive Practice, and Positive Youth Development.

The 2017 DYS Mathematics Instructional Guide was developed in collaboration with DYS teachers and content experts; we trust you will find it relevant and useful in planning rigorous and engaging instruction for youth in the DYS setting. Thank you for your commitment and dedication to providing youth in our care with a quality educational experience as they prepare to transition from DYS as young adults who are ready for their futures.

Sincerely yours,

Christine Kenney, Director of Educational Services
Massachusetts Department of Youth Services
Using these Materials and Resources

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DYS–2017 Mathematics Instructional Guide

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Acknowledgments

This resource emphasizes teaching and learning Mathematics, and is part of a series of instructional guides focusing on the content and delivery of educational services in Massachusetts Department of Youth Services (DYS) facilities across the Commonwealth. DYS Instructional Guides are one component of the Comprehensive Education Partnership’s Education Initiative, an education reform initiative supported by Commonwealth Corporation and the Collaborative for Educational Services.

All materials in this Guide align both with standards from the 2017 Massachusetts Curriculum Framework for Mathematics and with the Common Core State Standards for Mathematics.

The content within these pages has been developed through the efforts of talented and dedicated practitioners who have generously shared their expertise and best thinking about effective Mathematics instruction.

We especially want to recognize the Massachusetts Department of Youth Services, its students, and the educators and program staff who work every day to bring clarity and focus to the delivery of educational services within the DYS system.
Acknowledgments

The following individuals and organizations were instrumental in the development of this Guide:

**PROJECT COORDINATORS**

Michael Dialessi  
*Project Manager and Designer*

Laura Finn-Heafey  
*Director of Curriculum, Instruction and Assessment*  
*DYS Education Initiative*

Lucia Foley  
*Design Projects Coordinator, CES*

**CONTRIBUTORS**

**Curriculum Work Group Members:**

- Heidi Cahoon-McEwen  
  *SEIS Assistant Director*

- Diana DePaolis  
  *CommCorp Program Manager*

- Sonia Febres  
  *DYS Title I and Targeted Learning Supports Coordinator*

- Mary Murray  
  *Education Consultant for DYS*

- Kariña Monroe  
  *DYS Assistant Regional Education Coordinator*

- Lissa Picard  
  *SEIS Professional Development Coordinator*

- David Smokler  
  *DYS Associate Director, Education and Transition Services*

- Erin White  
  *DYS Instructional Coach, Western Region*

- Darnell Thigpen Williams  
  *DYS Associate Director of Professional Development*

**Phase 1 and 2 DYS Teachers:**

- Amanda Brooks-Clemeno
- Prabir Chandra
- Ron Perrot

**Administrative support provided by:**

- Betsy Bender
- Barbara Bridger
- Patti Matthews
- Joy May

We are grateful for the valuable contributions provided by the DYS Education Initiative staff, including Regional Education Coordinators, Assistant Regional Education Coordinators, Instructional Coaches, and Education and Career Counselors.
Acknowledgments (continued)

WRITERS

Mathematics Writing Team—Phases 1 and 2:

Michelle Bussiere, Writing Team Leader (Phase 1 and 2)
Michelle taught mathematics at South Hadley High School beginning in 1993. She has also worked with the Massachusetts Department of Elementary and Secondary Education on MCAS test development and was a member of the MCAS 10th-grade ADC Test Committee. Michelle also serves as a teacher-consultant in the Western Massachusetts Writing Project, and has presented professional development workshops for teachers and trainers throughout western Massachusetts on “Writing in Mathematics,” “How Learning Styles Can Impact Your Teaching” and “Unpacking the DYS Mathematics Instructional Guide.”

Derek Chace, Curriculum Writer
Derek taught middle school science and math at the E. N. White School in Holyoke. Previously, he worked at the William J. Dean Technical High School for two years where he taught mathematics, biology, physics, and chemistry. He received his M.S. in Physics from the University of Massachusetts, Amherst in 2005.

Kathryn Chace, Curriculum Writer
Kathryn taught middle school mathematics at the E. N. White School in Holyoke for the past eight years. She received her M.S. in Cognitive Psychology with a minor in Statistics from the University of Massachusetts, Amherst in 2006.

Scott Hsu, Curriculum Writer
Scott taught high school math at both South Hadley High School and the Pioneer Valley Chinese Immersion Charter School. He has pioneered a flipped classroom instructional model using self-produced instructional videos to teach his students. Scott served on the DYS Mathematics Instructional Guide writing team during its initial phase, and is currently living in New Delhi, India, where he is teaching at the American Embassy School.

Christophe Huestis, Curriculum Writer
Christophe has taught high school mathematics for sixteen years, first at White Oak School in Westfield, where he taught reading, writing, and mathematics to students with moderate to severe language-based learning disabilities. Currently, he teaches at Agawam High School where he leads the Mathematics Department as the Core Curriculum Facilitator; creating and implementing professional development, and writing curriculum. As a department leader, Christophe has experience with writing math curriculum and presenting on the topic of math instruction for LBLD populations.

Dianna Schulte McMenamin, Content Editor
Dianna has degrees in Biology and Geological Sciences and is the co-author of two books on geology. She has tutored and taught mathematics enrichment courses for grades K-12 at South Hadley Public Schools, and as a homeschooling parent of her three children.

Jeffrey Taylor
Jeffrey is a special education teacher at South Hadley High School where he also teaches algebra. Previously, he taught alternative mathematics classes and co-taught algebra, geometry, and pre-calculus at Berkshire Art and Technology Charter School where he also assisted with alignment of curricula to Common Core State Standards. Jeffrey graduated with a B.A. in History and Secondary Education. In 2011 he received his M.Ed. in Special Education from the University of Massachusetts, Amherst, where he served as the assistant to the editor of the Journal of Special Education Leadership, and assisted with special education research and data collection. In 2008, Jeffrey taught business, citizenship, communication, and information technology courses at Warren Comprehensive High School in London, England.
Acknowledgments (continued)

ABOUT THE WESTERN MASSACHUSETTS WRITING PROJECT

WMWP, a local site of the National Writing Project, is a university-school partnership based in the English Department of the University of Massachusetts Amherst that offers professional development and leadership opportunities for educators in Western Massachusetts designed to improve writing and learning for all students.

WMWP’s mission is to create a professional community where teachers and other educators feel welcomed to come together to deepen individual and collective experiences as writers and understanding of teaching and learning in order to challenge and transform practice.

WMWP’s aim is to improve learning in our schools—urban, rural and suburban. Professional development provided by the Western Massachusetts Writing Project values reflection and inquiry and is built on teacher knowledge, expertise, and leadership. Central to WMWP’s mission is the development of programs and opportunities that are accessible and relevant to teachers, students, and their families from diverse backgrounds, paying attention to issues of race, gender, language, class, and culture and how these are linked to teaching and learning.

Western Massachusetts Writing Project (WMWP)
Writing Team—Phases 2 and 3:

Jane Baer-Leighton, Team Leader and Project Coordinator
Jane is the WMWP Professional Development Coordinator and has been a teacher-consultant since 2003. She was an outside facilitator for DYS professional development in 2013-14 and a copy editor for the 2014 DYS English Language Arts Instructional Guide and 2016 DYS Science Instructional Guide. She was a co-facilitator for the WMWP Literacy Leadership Institute, which brought together school administrators and teacher-consultants to research and connect current ESE initiatives, including the Common Core standards, and create a coherent approach to professional development. Jane was an English teacher at Amherst Regional High School for 35 years.

Richard Dubuisson, Mathematics Content Specialist
Richard Dubuisson has over 20 years of experience as a high school math teacher, instructional coach, curriculum developer, and education consultant. He currently serves as Director of Program Design at Year Up, a national non-profit working to close the opportunity divide in this country. A native of Haiti, Richard is passionate about helping schools create and provide quality educational opportunities to under-served young adults to put them on a path to empowerment, life-long learning and economic self-sufficiency.

Johanna Greenough, Mathematics Content Specialist
Johanna Greenough began her teaching career as a Montessori teacher of first and second graders at a public Montessori school. After a couple years, she found her love of teaching upper elementary at the Montessori School of Northampton, where she has co-taught the fourth- through sixth-grade classroom for eight years. Over the past several years, she has developed a great passion for teaching mathematics, and this year she got the chance to teach math in her school’s brand new middle school. Johanna is currently pursuing a Master’s of Mathematics Teaching at Mount Holyoke College. She is constantly seeking ways to spread the joy and love of math. Johanna is a teacher-consultant for the Western Massachusetts Writing Project.
Acknowledgments (continued)

Nina Koch, Mathematics Content Specialist
Nina Koch taught mathematics and computer programming at Amherst Regional High School for 30 years. During that time, she developed the curriculum and programmed educational software for new courses in Transformational Geometry and Quantitative Reasoning. She also wrote a chapter included in “What’s Happening in Math Class” published by Teachers College Press in 1996. Nina has a special interest in number theory.

Karen Miele, Curriculum Writer
A WMWP teacher-consultant since 2010, Karen has served as the site’s Co-director of Outreach and as co-facilitator of the Summer Institute. She helped write the 2014 DYS English Language Arts Instructional Guide and 2016 DYS Science Instructional Guide. She has presented workshops in area schools on integrating the Common Core standards. As a 9th and 11th grade English teacher in the Grafton Public Schools, she helped revise Grafton High School’s English curriculum to align with the Common Core standards.

Bruce M. Penniman, Ed.D., Curriculum Writer and Copy Editor
Bruce served as Writing Team leader and project coordinator for the 2014 DYS English Language Arts Instructional Guide and the 2016 DYS Science Instructional Guide. He has been a WMWP teacher-consultant since 1994 and is currently site director. Bruce taught English and journalism in the Amherst Regional Schools for 36 years and served as a facilitator/coach in the ESE Model Curriculum Units project. He was 1999 Massachusetts Teacher of the Year and is the author of Building the English Classroom: Foundations, Support, Success (NCTE, 2009).

Jodi Stevens, Mathematics Content Specialist
A math and science teacher at Amherst Regional Middle School, Jodi began her career as a middle school science teacher in the late 1990s. She added middle school math to her teaching repertoire in 2010 and is currently teaching both subjects. For both her math and science students, she works to create curriculum that is inviting and engaging. Her goal is always to inspire students who love science and think math is very cool.

Staff Writer:
Diana DePaolis, Empower Your Future Program Manager
A former English teacher, Diana supports the Department of Youth Services Education and Workforce Development Initiative at Commonwealth Corporation. She previously taught ninth-grade English in the Somerville Public Schools and worked as a content specialist at the Achievement Network where she created Common Core aligned ELA assessments and provided professional development for teachers and coaches.
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DYS Instructional Guide Purpose

The 2017 DYS Mathematics Instructional Guide is designed to be used as a “go to” resource that helps Massachusetts Department of Youth Services (DYS) teachers in planning robust curriculum, assessments, and instruction that maximize opportunities for youth to experience academic success.

The Guide is organized in chapters and features a Scope and Sequence for each subject, exemplar units and adaptations for both long- and short-term programs, and resources that support teachers in planning for instruction. Curriculum, planning, assessments, and instruction are organized around key themes and essential learning outcomes. All educators are responsible for creating rigorous and personalized learning opportunities and multiple access points to the curriculum. As well, teachers plan and implement instruction with an understanding of and appreciation for the richness of diversity within the student population.

The Common Core State Standards (CCSS) give teachers a strengthened academic foundation from which to set high expectations for our youth, and the curricular guidance to help youth experience academic success so they can learn the skills they need to be “future ready” when they transition from DYS to their communities (“About the Standards”).

By aligning the DYS Instructional Guides with the CCSS and the 2017 Massachusetts Curriculum Framework for Mathematics and integrating these standards into our instruction, we support college and career readiness and provide the bridge that connects academic learning to the real world. Teachers are encouraged to access the website links located in the Appendix at the end of this Guide to learn more about the Common Core and the MA Mathematics Framework.

These standards and shifts are unpacked for teachers in Chapter 1 and their relationship to curriculum, planning and instruction is developed in Chapter 2. Chapter 3 emphasizes proper pedagogical practices for teaching mathematics in DYS schools that includes building mathematics knowledge and skills, student engagement, and assessment. The curriculum Scope and Sequences, located in each subject chapter, reflect a careful focus on Emphasized Standards for each unit that support students’ skill development, ongoing learning and mastery.

Who Are Our Youth?

Data and statistics do not tell the whole story of our youth. The DYS youth population is diverse in every aspect including educational levels, background knowledge and experiences, interests, aspirations, learning styles, multiple intelligences, and social-emotional strengths and challenges. Our youth are readers, writers, thinkers, musicians, mathematicians, artists, athletes, students, employees, and members of the community. Our youth are family members. Some of our youth are parents. Our youth are life-long learners. Some have done well in school and will use our classes to build and expand their success as learners. Others have experienced academic challenges or frustrations in the past. Our youth learn best when actively engaged and connected to real-world experiences and contexts. Our youth are all youth.
DYS Education Mission

DYS seeks to provide a comprehensive educational system that meets the needs, experiences, and goals of our youth. Through collaboration with local schools, community-based organizations, families, and other resources, we provide a personalized student plan that is standards-based and focuses on literacy and numeracy skills; education, employment and training opportunities; and transitions to the community and the workforce.

DYS Education Programs

DYS Educational Services strives to meet all Massachusetts Department of Elementary and Secondary Education (ESE) Standards and Indicators for Effective Teaching Practice: curriculum planning and assessment, teaching all students, family and community engagement, and professional culture. These standards drive our intentional approach in providing high-quality educational services for youth. As well, we adhere to ESE policies and procedures including requirements for time on learning, highly qualified educator certification, and teacher evaluation.

Educational programs operate under contract with DYS. Accountability standards have been put in place to ensure greater standardization of the educational programming across the system. While size, type, location, security levels, and other factors vary a great deal among DYS programs across the Commonwealth, all DYS settings are

DYS Program Types

Detention programs primarily house youth who have been charged with a criminal offense and are being held on bail awaiting court action. These units may also house youth who are committed and are awaiting placement in another facility or program, or who are in the process of revocation from a community placement.

Assessment programs are designed to evaluate the needs of newly committed youth. The Department administers several risk/needs assessments in the areas of mental health and substance abuse and educational testing. This information, as well as information about families, any prior contact in the juvenile justice system, and the offense history informs placement decisions. The typical length of stay in an assessment program is 30-45 days.

Treatment programs offer supervision and treatment, and the most appropriate placement may be determined by the Department’s assessment.

Short-term treatment: The average length of stay in this type of program is 90 days, but placement may be extended to 180 days.

Long-term treatment: The average length of stay is 8-12 months, although some youth may stay longer.

Revocation programs serve youth who have been released from a DYS Treatment Program and are having difficulty adjusting to the community. They have broken the conditions of their earlier release and are therefore revoked back into the care and custody of DYS (Massachusetts).
united by shared principles, guidelines, professional development, curriculum materials, and coaching. Educational programming operates on a 12-month school year and provides a minimum of 27.5 hours of instructional services per week.

**Vision of Integrated Service Delivery**

The Special Education in Institutional Settings (SEIS) program, an ESE program, delivers special education services for approximately 55% of the DYS student population in residential settings. DYS and ESE have adopted an integrated service delivery approach to guide our comprehensive educational efforts. The phrase “integrated service delivery” reflects our core belief that youth need coordinated supports in order to make progress that has a lasting, positive impact on their futures. Evidence of this core belief is found across many of our established practices, most notably such structured collaborative practices as Learning Teams and the Agency Coordination Process. Integrated service delivery is not a separate strategy. Integration of services informs all aspects of the teaching and learning process so we ensure that we are collaboratively meeting the educational needs of each youth in our setting.

Students and teachers are also supported in some programs by Literacy Specialists funded through Title I, a federal grant program. English Language Learners (ELLs) are supported by teachers, across all educational programs, who have been trained in providing instruction to that population. Through the collaborative work of all personnel, a continuum of services is planned for and implemented, responding to individual needs, and allowing for access to the general education curriculum in the least restrictive environment.

**A Personalized Approach**

We are committed to providing a personalized approach for all youth to education and transition planning so they are ready for their futures.

The DYS Education Initiative describes a personalized approach as a learning process between students, educators, and other caring adults in which students are helped to assess their own strengths and aspirations, plan for and make demonstrated progress toward their own purposes, and work cooperatively with others to accomplish challenging tasks. With the individually tailored support and guidance of caring adults, students evidence their explorations, accomplishments, and work by demonstrating learning against clear and relevant standards (Clarke; Rennie Center).

Youth placed in DYS programs require a personalized approach to all aspects of their growth and development. As educators, it is our collective responsibility to both build on the strengths and meet the needs of each student who enters our classrooms. All DYS youth are placed into a course of study that best meets their individual needs.

---

**Courses of Study: Areas of Concentration**

**High School:** Students who are concentrating on obtaining a high school diploma are placed in classes in accordance with the graduation requirements and their educational records from their sending districts.

**High School Equivalency:** Students who have met DYS policy requirements for pursuing their high school equivalency credential are placed in core content classes identified in their practice tests as requiring additional study.

**Postsecondary:** Students who have already earned a high school diploma or its equivalency are eligible to enroll in college coursework or other postsecondary programs as articulated in their transition plans.

**Career Readiness:** Students who have already earned a high school diploma or equivalency and are actively preparing for college or postsecondary opportunities, or students who are not actively pursuing college, or students who are 18 or older and have formally withdrawn from school may pursue the career opportunities articulated in their transition plans.
The DYS Approach to Curriculum, Planning, and Instruction

DYS teachers use research-based instructional models to plan relevant and rigorous curriculum and instruction to address the variability of learners. These models are the very core of our instructional pedagogy. Understanding by Design (UbD) and Universal Design for Learning (UDL) intersect with Differentiated Instruction (DI) and the frameworks for Culturally Responsive Practice (CRP), Positive Youth Development (PYD) and Empower Your Future (EYF) to serve as a strong and effective foundation for curriculum design so that teachers may best meet the myriad of learning needs of DYS youth. Following is a brief description of each of these models.

Understanding by Design (UbD)

DYS teachers use the UbD model to develop instructional units and lessons. The principles of UbD guide DYS teachers to ask, “What do I want my students to know, understand, and be able to do at the end of this lesson or unit?” They determine at the onset of planning what the “desired result” will be based on state standards and learning objectives. Next, they ask, “How will students demonstrate their learning?” And finally they ask, “What learning experiences can I plan that support these learning goals and outcomes?” The framework includes three stages of curriculum development:

Stage 1: Identify desired outcomes and results.
Stage 2: Determine what constitutes acceptable evidence of competency in the outcomes and results (assessment).
Stage 3: Plan instructional strategies and learning experiences that bring students to these competency levels (Wiggins and McTighe).

Universal Design for Learning (UDL)

Universal Design for Learning is defined as “a set of principles for curriculum development that gives all individuals equal opportunities to learn. UDL provides a blueprint for creating instructional goals, methods, materials, and assessments that work for everyone—not a single, one-size-fits-all solution but rather flexible approaches that can be customized and adjusted for individual needs” (CAST).

At the start of the UDL planning process, when considering curriculum design, the teacher incorporates multiple means of:

Engagement
The teacher considers “how learners get engaged and stay motivated” and how he or she will “stimulate interest and motivation for learning.”

Representation
The teacher considers how and in what ways the information and content will be presented so that individual learner needs may be met.

Action and Expression
The teacher considers what tasks will be in the lesson or unit offered to students and how those tasks will be varied so that students with different needs can express what they know.

Teachers apply the principles of UDL at the start of the curriculum planning process to provide effective instruction for a wide range of learners. Within that context, teachers monitor student learning and differentiate instruction further depending upon the variability of learner needs.

Differentiated Instruction (DI)

All youth benefit most from instruction that is differentiated to fit their specific learning needs and maximize their learning potential. Differentiated
instruction “applies multiple approaches to instruction that identify and integrate differences in culture, family values, and academic background to help teach students of varying learning levels. Teachers need to assess students’ readiness, preferences, and interests, and be responsive to these needs.” Teachers differentiate instruction by content, process and product, employing multiple instruction and assessment strategies that work to create ease of access to the general curriculum (“What’s Different”).

**Culturally Responsive Practice (CRP)**

Understanding the identities and experiences of youth and the worlds that have shaped them is another form of differentiation known as Culturally Responsive Practice.

Culturally responsive teaching involves linking the curriculum to the lives of our youth in authentic and meaningful ways for the purpose of helping them achieve success. Culturally responsive teaching involves reflecting on the ways in which we interact with our youth and they interact with one another to form positive and affirming experiences. To be culturally responsive educators means getting to know our youth and learning how their experiences and identities have shaped the way they see the world. It involves developing an awareness of how teachers view their own world and how this influences their way of teaching. When we build connections between the worlds of our youth and our own and use these connections to inform our teaching, our youth can see themselves as active and valued participants in the learning community.

**Positive Youth Development (PYD)**

Underlying all aspects of the approach in working with youth in DYS is Positive Youth Development.

This model focuses on the positive attributes young people need to make a successful transition to adulthood. The PYD framework revolves around the cognitive, emotional, and social needs of a young person. A strong focus on these aspects of PYD provides effective guidance for the goals and plans for each youth’s successful re-entry into the community. These include:

1. Focusing on each youth’s strengths and personal assets
2. Providing opportunities for youth empowerment and leadership
3. Cultivating community partnerships and supports that assist youth in moving successfully through the continuum of care

Learning occurs when young people perceive that they are valued as members of the learning community, that teachers believe in them, and that they are expected to succeed. Teachers build caring relationships that are informed by knowledge of cultural backgrounds, previous experiences, and personal strengths of all youth.

Building strong relationships with caring adults, being held to high expectations, and participating in growth opportunities are also cornerstones to a PYD approach that leads to all youth achieving positive outcomes.

**Empower Your Future (EYF)**

The Empower Your Future Initiative is a project-based curriculum and a future-focused approach for developing youth voice in education, career, and transition planning. DYS has included EYF within Goal 3 of the DYS Strategic Plan:

Youth will sustain the gains they made while in DYS custody through improved transition planning and continuing community supportive partnerships.
By teaching self-advocacy skills, EYF helps youth sustain gains made in school and involves them in planning their own futures. When youth voice is integrated into planning, the Education and Career Counselors can ensure that gains made by youth are communicated to educators and other staff. Knowledge of youths’ strengths, interests, and needs serves to connect services across the DYS continuum of care.

The EYF curriculum is standards-based and designed to help youth develop the academic/technical, workplace readiness, and personal/social competencies outlined in the Massachusetts Career Development Benchmarks. The EYF curriculum reflects the Massachusetts Department of Elementary and Secondary Education’s (ESE) approach to college and career planning. The Empower Your Future Initiative provides curriculum specifically tailored for each program across the continuum of care: detention, assessment, treatment and revocation; and in the community through Bridging the Opportunity Gap programs (Empower Your Future).

Throughout this Guide you will find this icon to indicate connections to Empower Your Future (EYF). Following each unit you will find a detailed description of those EYF connections.

Works Cited

www.doe.mass.edu/frameworks


www.renniecenter.org/topics/alternative_education.html.

“What’s Different about Differentiated Instruction?” Dreambox Learning. 2014. 
www.dreamboxlearning.com/blog/whats-differentiated-instruction.

# 1 High Standards and Effective Practice

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High Standards and Effective Practice
For Mathematics Instruction

High-Quality Education
The students served by Department of Youth Services (DYS) programs are diverse and face many challenges, but all deserve access to courses that are created with high standards and taught with effective practice.

DYS educational programs provide a rigorous curriculum aligned with the Massachusetts Curriculum Frameworks and the Common Core State Standards (CCSS) for Mathematics and is designed with a personalized approach to instruction to meet the needs of all learners.

The DYS Mathematics Instructional Guide was designed to aid teachers in understanding the CCSS and the Massachusetts Curriculum Frameworks, as well as guide them in their planning of units, lessons, and assessments.

Aligned with State Standards
The Mathematics Scope and Sequence and the exemplar units and sample lessons in this Guide are grounded

“There is more than a balance between the classroom—both are occurring with intensity.”

INSTRUCTIONAL SHIFTS IN MATHEMATICS

<table>
<thead>
<tr>
<th></th>
<th>Focus</th>
<th>Teachers significantly narrow and deepen the scope of how time and energy is spent in the math classroom. They do so in order to focus deeply on only the concepts that are prioritized in the standards.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Coherence</td>
<td>Principals and teachers carefully connect the learning within and across grades so that students can build new understanding onto foundations built in previous years.</td>
</tr>
<tr>
<td>3</td>
<td>Fluency</td>
<td>Students are expected to have speed and accuracy with simple calculations; teachers structure class time and/or homework time for students to memorize, through repetition, core functions.</td>
</tr>
<tr>
<td>4</td>
<td>Deep</td>
<td>Students deeply understand and can operate easily within a math concept before moving on. They learn more than the trick to get the answer right. They learn the math.</td>
</tr>
<tr>
<td></td>
<td>Understanding</td>
<td></td>
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<tr>
<td>5</td>
<td>Applications</td>
<td>Students are expected to use math and choose the appropriate concept for application even when they are not prompted to do so.</td>
</tr>
<tr>
<td>6</td>
<td>Dual Intensity</td>
<td>Students are practicing and understanding. There is more than a balance between these two things in the classroom—both are occurring with intensity.</td>
</tr>
</tbody>
</table>

Source: www.engageNY.org
consistent framework to prepare students for college and
the workforce. For more information about CCSS:

**SEE:** Common Core State Standards Initiative
www.corestandards.org

Education Next
educationnext.org/few-states-set-worldclass-standards

CCSS developers sought to create “fewer, clearer, higher”
standards than previously existed in Massachusetts
or other states. The new framework is different in
several important ways, including key curricular “shifts”
summarized in the table on p. 1.1.1. The standards stress
intensive focus (fewer topics are taught in greater depth),
linking back (with previous concepts), and mathematical
modeling. For a detailed explanation of these “shifts,” visit
the EngageNY website.

**SEE:** Engage NY
www.engageNY.org

In addition, “The 2017 Massachusetts Curriculum
Framework for Mathematics reflects improvements to the
prior Framework. ... The 2017 standards draw from the
best of prior Massachusetts standards, [the Common
Core State Standards initiative], and represent the input

### FOCUS OF INSTRUCTIONAL TIME IN MATHEMATICS COURSES (High School Level)

#### ALGEBRA 1

1. Deepen and extend understanding of linear and exponential relationships
2. Contrast linear and exponential relationships with each other and engage
   in methods for analyzing, solving, and using quadratic functions
   *(NOTE: Previously an Algebra 2 standard)*
3. Extend the laws of exponents to square and cube roots
4. Apply linear models to data that exhibit a linear trend

#### GEOMETRY

1. Establish criteria for congruence of triangles based on rigid motions
2. Establish criteria for similarity of triangles based on dilations and proportional reasoning
3. Informally develop explanations of circumference, area, and volume formulas
4. Apply the Pythagorean theorem to the coordinate plan
5. Prove basic geometric theorems
6. Extend work with probability

#### ALGEBRA 2

1. Extend work with probability
2. Expand understandings of functions and graphing to include trigonometric functions
3. Synthesize and generalize functions and extend understanding of exponential functions
to logarithmic functions
4. Relate data display and summary statistics to probability and explore a variety of data
collection methods

of hundreds of the Commonwealth’s K-12 and higher education faculty. The 2017 standards present the Commonwealth’s commitment to providing all students with a world-class education.”

Key changes in the revised Framework include:

- Revised language to clarify the standards, including the incorporation of footnotes into the standards, definitions of key terms, and examples to specify expectations for students
- A revised high school section with an explanation of the alignment between the Conceptual Category Standards and the Model High School course standards in the two model pathways (Traditional and Integrated)
- New and revised Guiding Principles

These changes and other important features from the 2017 Massachusetts Curriculum Framework for Mathematics will be highlighted in this chapter.

The Framework can be found on the ESE website at:

SEE: Massachusetts Curriculum Frameworks
www.doe.mass.edu/frameworks

Model Traditional Pathways

The DYS Mathematics Curriculum Guide will follow the Model Traditional Pathways for Algebra 1, Geometry, and Algebra 2. It is important to recognize that the content standards for each pathway are organized in Conceptual Categories with domains and clusters, instead of a list of standards and strands. An introduction to the coding of content standards can be found on page 103 of the 2017 Mathematics Curriculum Framework.

When examining the content standards for each course in the Model Traditional Pathway, teachers will also notice that many content standards have been revised to clarify meaning and student expectations. The revisions also incorporate footnoted explanations into these standards and provide definitions of key terms.

The focus of instructional time in Algebra 1, Geometry, and Algebra 2 courses has also changed as noted. The focus of instructional time within the Model Traditional Pathways is listed by course in the table on the preceding page (p. 1.1.2). For a more detailed explanation, please see pp. 105, 112, and 118 respectively in the 2017 Massachusetts Curriculum Framework for Mathematics.

Eight Guiding Principles

The Eight Guiding Principles are the philosophical statements that underlie the Standards for Mathematical Practice in Massachusetts. These principles should guide the design and evaluation of mathematics programs. Each teacher should reflect on these principles daily to inform his/her approach to teaching curricula in order to meet the individual learning needs of each student. For detailed explanations of these eight principles, please download the 2017 Massachusetts Curriculum Framework for Mathematics using the ESE website link provided.

Guiding Principle 1

Educators must have a deep understanding of the mathematics they teach, not only to help students learn how to efficiently do mathematical calculations, but also to help them understand the fundamental principles of mathematics that are the basis for those operations. Teachers should work with their students to master these underlying concepts and the relationships between them in order to lay a foundation for higher-level mathematics, strengthen their capacity for thinking logically and rigorously, and develop an appreciation for the beauty of math.

Guiding Principle 2

To help all students develop a full understanding of mathematical concepts and procedural mastery, educators should provide them with opportunities to apply their learning and solve problems using multiple methods, in collaboration with their peers and independently, and complemented by clear explanations of the underlying mathematics.

Guiding Principle 3

Students should have frequent opportunities to discuss and write about various approaches to solving problems, in order to help them develop and demonstrate their mathematical knowledge, while drawing connections between alternative strategies and evaluating their comparative strengths and weaknesses.
Guiding Principle 4
Students should be asked to solve a diverse set of real world and other mathematical problems, including equations that develop and challenge their computational skills and word problems that require students to design their own equations and mathematical models. Students learn that with persistence they can solve challenging problems and be successful.

Guiding Principle 5
A central part of an effective mathematics curriculum should be the development of a specialized mathematical vocabulary including notations and symbols, and an ability to read mathematical texts and information from a variety of sources with understanding.

Guiding Principle 6
Assessment of student learning should be a daily part of a mathematics curriculum to ensure that students are progressing in their knowledge and skill, and to provide teachers with the information they need to adjust their instruction and differentiate their support of individual students.

Guiding Principle 7
Students at all levels should have an opportunity to use appropriate technological tools to communicate ideas, provide a dynamic approach to mathematics concepts, and to search for information. Technological tools can also be used to improve efficiency of calculation and enable more sophisticated analysis, without sacrificing operational fluency and automaticity.

Guiding Principle 8
Social and emotional learning can increase academic achievement, improve attitudes and behaviors, and reduce emotional distress. Students should practice self-awareness, self-management, social awareness, responsible decision-making, and relationship skills, by, for example, collaborating and learning from others and showing respect for others’ ideas, applying the mathematics they know to make responsible decisions to solve problems, engaging and persisting in solving challenging problems, and learning that with effort, they can continue to improve and be successful (Mass. Framework, p. 17).

Eight Standards for Mathematical Practices
In addition to the Focus of Instructional Time in Mathematics Courses and the Eight Guiding Principles for Mathematics Programs in Massachusetts, mathematics teachers must also reflect on the Standards for Mathematical Practice (SMP). The 2017 Massachusetts Curriculum Framework for Mathematics (pp. 19-21) explains that “the Standards for Mathematical Practice describe expertise that mathematics educators at all levels should seek to develop in their students. The SMP describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years...,” and, “Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.”

The SMP describe eight skills that students will need to become mathematically proficient. A list of these practice standards with descriptions is found on p. 1.1.5 and is also included at the bottom of the Scope and Sequence charts,
STANDARDS FOR MATHEMATICAL PRACTICE (SMP)

Mathematically proficient students will...

1. **Make sense of problems and persevere in solving them.** Students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They check their answers to problems using a different method and continually ask themselves, “Does this answer make sense?”

2. **Reason abstractly and quantitatively.** Students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically, and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved.

3. **Construct viable arguments and critique the reasoning of others.** Students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counter examples. They justify their conclusions, communicate them to others, and respond to the arguments of others.

4. **Model with mathematics.** Students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another.

5. **Use appropriate tools strategically.** Students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software.

6. **Attend to precision.** Students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context.

7. **Look for and make use of structure.** Students look closely to discern a pattern or structure. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. Students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another.

8. **Look for and express regularity in repeated reasoning.** Students notice if calculations are repeated, and look both for general methods and for shortcuts. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

*Note: These standards have been edited to reflect a high school focus.*

2017 Massachusetts Curriculum Framework for Mathematics (pp. 19-21), www.doe.mass.edu/frameworks
located on pp. 4.2.2 to 4.3.2 in Chapter 4 for Algebra 1, pp. 5.2.2 to 5.2.3 in Chapter 5 for Geometry, and pp. 6.2.2 to 6.2.3 in Chapter 6 for Algebra 2. All eight of these standards for mathematical practice are important and relevant in the planning of all mathematics units. Teachers will not incorporate all SMP into each lesson; however, by the end of a unit of study each of the SMP should be addressed.

To assist teachers in the planning process, an example of the integration of each Standard of Mathematical Practice is provided below from the fall exemplar units in Algebra 1, Geometry, and Algebra 2.

“Linear Modeling” is the exemplar unit for the Fall Season of Algebra 1. The following examples explain how four of the Standards for Mathematical Practice have been emphasized.

**SMP.1:** Make sense of problems and persevere in solving them.

**SMP.2:** Reason abstractly and quantitatively.

**SMP.4:** Model with mathematics.

**SMP.7:** Look for and make use of structure.

In Lesson 1, students practice SMP.1 as they learn to distinguish between relationships that are linear and those that are not. In one real-world scenario, they initially analyze the relationships between the number of three types of window panes (corner, side, and middle) and the overall area of square and rectangular windows. In succeeding lessons, students draw diagrams of important features, discuss relationships among variables, create data tables, graph data, and create equations. They explain the patterns they observe and how equations, verbal descriptions, tables and graphs correspond with each other. At several points during each activity, students check and discuss their answers with classmates and reflect on the relationships among the variables they are observing.

Students are introduced to SMP.2 in Lesson 2 when they are given a set of equations and a set of situations and are asked to match the situation to the equation that it represents. At this point, students learn to make sense of variables and their relationships in real-world problem situations. As students begin to understand the meaning of the y-intercept and slope, they will discover how they can represent linear situations symbolically, explain the significance of a specific slope and y-intercept in a situation, create a coherent representation of a problem, and understand how their equations can be used to make predictions.

Throughout the entire unit, students engage in SMP.4. Each lesson focuses on a real-world linear problem that students are asked to solve. The Performance Task, the culminating assessment for the unit, asks students to determine whether it is more economically feasible for a friend to buy a newer car or maintain their current car. The students identify important one-time as well as ongoing costs of maintaining a vehicle and translate their relationships into diagrams, tables, graphs, equations. They synthesize a recommended course of action from their mathematical analyses and present their conclusions to their peers.

Students practice SMP.7 in Lesson 1 as they recognize the slope intercept form of a linear equation, $y = mx + b$ (or $y = ax + b$), with a constant rate of change, in contrast to non-linear equations that contain exponents or inverses or other non-linear functions. In later lessons, students learn to identify where in a linear equation the slope and y-intercept are located, deduce a line’s slope from coordinates of two points, and interpret the slope as the relationship between the relative changes in two variables.

The fall exemplar unit in Geometry is “Parallel and Perpendicular Lines.” The following examples illustrate how three Standards for Mathematical Practice are used.

**SMP.3:** Construct viable arguments and critique the reasoning of others.

**SMP.5:** Use appropriate tools strategically.

**SMP.6:** Attend to precision.

SMP.3 is practiced during Lesson 4, when students are introduced to the idea of proofs and are required to use deductive reasoning and logic to prove the properties of parallel lines. Students will continue to work with proofs throughout the unit by sharing their lines of reasoning
and critiquing the explanations of their classmates. When students present their performance assessments, they will show their classmates optical illusions that appear to have lines that are not parallel, when the lines in fact are parallel. Students will need to justify their conclusions that the lines in their illusion are parallel. Classmates will listen to the presenter’s reasoning to decide whether or not the presenter’s justification makes sense.

SMP .5 is practiced in Lesson 2, when students are shown how to use a protractor to measure angles and are given time to practice using one. The standard continues to be addressed in the Performance Assessment where students will use rulers and protractors to make their optical illusions. They will need to determine how best to use the mathematical tools at their disposal in order to prove that their lines are parallel.

Throughout the unit, students will engage with SMP .6. Specifically, students will use proper mathematical vocabulary when discussing types of angles in Lesson 2 and will continue doing so throughout the remaining lessons of the unit. When creating proofs and explaining why their performance assessments have parallel lines, students must use precise vocabulary and clearly communicate their line of reasoning to their classmates.

Furthermore, in order to classify angles, students must be able to precisely measure angles with their protractors. “Quadratic Functions” is the exemplar unit for the fall season of Algebra 2. The eighth standard is highlighted.

SMP 8: Look for and express regularity in repeated reasoning.

SMP 8 asks students to make inferences. Students look at repeated examples and draw general conclusions from the evidence. In Lesson 4 in the quadratics unit, students are investigating the parameters (A, H, and K) of vertex form and reaching their own conclusions about their roles in quadratic functions.

Building Mathematical Understanding: A Balanced Approach


The standards strategically develop students’ mathematical understanding and skills. When students are first introduced to a mathematical concept they
explore and investigate the concept by using concrete objects, visual models, drawings or representations to build their understanding.

In the early grades they develop number sense and work with numbers in many ways. They learn a variety of strategies to solve problems and use what they have learned about patterns in numbers and the properties of numbers to develop a strong understanding of number sense, decomposing and composing numbers, and the relationship between addition and subtraction, and multiplication and division. In calculations, they are expected to be able to use the most efficient and accurate way to solve a problem based on their understanding and knowledge of place value and properties of numbers.

Students reach fluency by building understanding of mathematical concepts (this lays a strong foundation that prepares students for more advanced math work) and by building automaticity in the recall of basic computation facts (addition, subtraction, multiplication, and division).

As students apply their mathematical knowledge and skills to solve real-world problems, they also gain an understanding of why mathematics is important throughout our lives.

The Standards of the CCSS and the Massachusetts Frameworks help inform a teacher’s practice and define what a student should know. They DO NOT supplant or replace a teacher’s practice, define how teachers should teach, limit all that can be taught in the classroom, explain interventions for students who perform above or below grade-level, who have special needs or who may be English Language Learners (ELLs). The teacher remains the key decision-maker in the classroom and the one responsible for delivering a high-quality education to students.

SEE: “What the Mathematics Curriculum Framework Does and Does Not Do”
2017 Massachusetts Curriculum Framework for Mathematics
www.doe.mass.edu/frameworks

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Curriculum Planning and Instruction
Goal-Driven Units with Access for All

Introduction

This Guide includes nine exemplar units: one for each of the “seasons” of Algebra 1, Geometry, and Algebra 2. All exemplars have been developed using the DYS unit template, which is based in part on Grant Wiggins and Jay McTighe’s “Understanding by Design” (UbD) model of backward planning.

Backward planning is, in essence, the simple and sensible idea that curriculum development should begin with identifying the Desired Results of a course of study and working backward from those goals to the Assessment Evidence that will determine whether they have been met and ultimately to the Learning Plan that will move students toward achieving them. The DYS unit template includes these three planning stages, and their various sections are annotated below as an aid to understanding and curriculum development.

### DYS UNIT TEMPLATE: STAGE 1 | MATHEMATICS GUIDE

<table>
<thead>
<tr>
<th>DESIRED RESULTS</th>
<th>EMphasized Standards (Content and College and Career Readiness Anchor Standards)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>The Scope and Sequence charts for each subject identify one or more Emphasized Standards from the Massachusetts Curriculum Framework for Mathematics for each curriculum season; an additional standard (or more) may be included if it will truly be a focus of instruction. The targeted standards must all be assessed in the unit.</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th></th>
<th>Essential Question(s) (Open-ended questions/concepts that lead to deeper thinking and understandings)</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>The Scope and Sequence charts also provide Essential Questions for each season. Each unit should include one or more of these questions (modified as needed), plus unit-specific questions as appropriate. Essential Questions should be open-ended and spur inquiry, not lead to set answers.</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th></th>
<th>Transfer Goal(s) (How will students apply their learning to other contexts?)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Each exemplar unit includes examples of Transfer Goals appropriate to the Emphasized Standards, Essential Questions, and themes of the season. Note that the goals provided are at the higher levels of Bloom’s taxonomy. Each unit should include similar long-range, college- and career-ready goals.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Learning and Language Objectives (Mastery Objectives)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Know: factual knowledge, key vocabulary</td>
<td>Understand: connections to essential concepts and contexts</td>
</tr>
<tr>
<td>Know objectives include relevant facts, background information, general academic vocabulary, and the terminology needed for success in the unit. A helpful rule of thumb: knowledge items could be assessed on a quiz.</td>
<td>Understand objectives, which should always be expressed in complete sentences, are the real take-aways of the unit, the “big ideas” that emerge from examining the Essential Questions through particular content. Understandings are not statements of fact.</td>
</tr>
<tr>
<td>Do: application, demonstration of knowledge, understandings</td>
<td>Do objectives represent the abilities that students must possess or develop to fulfill the expectations of the unit. Sometimes referred to as “process knowledge,” these often include specific skills such as citing evidence or formulating a claim.</td>
</tr>
</tbody>
</table>


Stage 1

Starting at the end may seem counter-intuitive, but the backward planning process works like a GPS: setting the destination comes first, then determining the route. Emphasized Standards establish the broad aims for all units, but the UbD approach to goal-setting also places high value on “big ideas” represented by Essential Questions and Understandings as well as on long-term Transfer Goals. Deciding what students should know and be able to do as a result of instruction is important, as always, but considering how they will make meaning from the unit and apply what they have learned in real life is crucial. Developing these higher-order goals is a thoughtful way of anticipating the challenging but legitimate student question, “Why do we have to learn this?” A teacher who sets authentic, relevant, thought-provoking learning goals always has a satisfactory answer.

Stage 2

The most critical step in the backward planning process is designing the assessments that will provide evidence of student learning. In the UbD model, the principal assessment is a Performance Task that serves as the culminating experience of the unit (or the entire season). Unlike a traditional Summative Assessment such as a test, a Performance Task asks students to transfer their learning to a new, authentic problem. Wiggins and McTighe recommend the “GRASPS” method of creating authentic Performance Task scenarios:

- **Goal**: Establish the goal, problem, challenge, or obstacle in the task.
- **Role**: Define the position or job of the students in the scenario.
- **Audience**: Identify the target audience, client, or constituency within the scenario.
- **Situation**: Set the context of the scenario. Explain the situation.
- **Product**: Clarify what the students will create and why they will create it.
- **Standards**: Provide students with a clear picture of success by issuing rubrics to the students or developing them with the students.

(Adapted from McTighe and Wiggins 172)

Because it is the direct result of instruction in the unit, the Performance Task actually drives the Learning Plan, as explained in the next section. But each unit should
also include several related assessments that enable the teacher—and the students—to monitor progress toward the performance goals.

One or more Pre-Assessments administered at the start of a unit can serve to activate prior knowledge, stimulate interest in the topic, and establish a baseline of skills. Pre-Assessments can range from low-stakes writing and informal group tasks to scaled-down versions of the culminating Performance Task, but they should be constructed to allow students to demonstrate what they know and can do, not highlight their shortcomings. Formative Assessments placed strategically throughout the unit can serve not only as checks on students’ acquisition of knowledge, understanding, and skills, but also as the means for developing their capabilities. A well-designed

### DYS UNIT TEMPLATE: STAGE 3 | MATHEMATICS GUIDE

#### Universal Design for Learning/Access for All

(A multiple means of representation, action and expression, and engagement)

A key consideration in curriculum design is how students with multiple learning styles and needs can gain access and succeed. Universal Design for Learning (UDL) calls for multiple means of representation, expression, and engagement (the what, how, and why of learning, respectively). Differentiation of instruction, including technology and arts integration, and the DYS emphasis on Positive Youth Development and Culturally Responsive Practice, provide avenues for access to all students; accommodations and modifications can be used to support students with special needs.

#### Literacy and/or Numeracy across Content Areas

(Reading, Writing, Speaking, Listening, and Language)

Reading, writing, speaking, listening, language, and numeracy are at the heart of all learning, and the math standards stress the development of literacy and numeracy skills, especially development of students’ knowledge and skills through written expression and modeling. Each unit should provide opportunities for reinforcement of general literacy and numeracy capacities and habits of mind. Each course introduction and exemplar unit introduction offers suggestions for ongoing skills development activities.

#### Resources

(Texts, materials, websites, etc.)

This section should list all print and non-print materials students will use, plus teaching resources such as handouts and discussion protocols, including publication information and/or URLs.

#### Outline of Lessons

(tasks and activities to support achievement of learning objectives)

**Introductory** (Build upon background knowledge, make meaning of content, incorporate ongoing Formative Assessments)

The Introductory lesson(s) should facilitate engagement in the unit by fostering connections to students’ previous knowledge, experiences, and interests. The Introductory section should also preview the goals and outcomes of the unit and include one or more Pre-Assessment activities.

**Instructional** (Stimulate interest, assess prior knowledge, connect to new information)

The Instructional lessons should include a well-organized sequence of activities designed to sustain students’ interest and enhance their knowledge, understanding, and skills. These lessons may be designed most effectively by working backward from the culminating Performance Task(s), ensuring that everything students must learn in order to succeed on the task is actually taught during the unit.

**Culminating** (Includes the Performance Task, i.e., summative assessment—measuring the achievement of learning objectives)

The Culminating lessons should provide a process for completing the Performance Task(s), with opportunities for peer and teacher feedback, revision, and sharing. This sequence should also provide occasions for revisiting the Essential Questions and reflecting on what has been learned.
Formative Assessment is a mini-task that moves students toward successful completion of the Performance Task by building a particular capacity—such as formulating a claim or citing relevant evidence.

**Stage 3**

The final step in unit development is creating the sequence of lessons that constitute the Learning Plan. **Sequence** is the key word at this stage of the process. Lessons should not be just a series of “interesting activities” or merely based on moving to the next topic in the textbook. Rather, they should be organized into introductory, instructional, and culminating experiences that foster continuous progress toward the performance goals. The best-designed lessons focus on what the students will learn, not what the teacher will teach, and they include mini-tasks that serve as Formative Assessments, as noted in the Learning Plan (see Stage 3 table on p. 2.1.3).

The DYS lesson plan template calls for each lesson to have six parts: Do Now, Hook, Presentation, Practice and Application, Review and Assessment, and Extension.

Normally, all of these aspects, which mimic the arc of the unit plan template from introduction to culmination, occur within a single class period. However, in some of the exemplar units there are lessons that extend over two or even more days to allow for in-depth engagement with a topic and/or extensive practice and discovery. In these lessons, the Practice and Application part of the lesson might begin on Day 1 and conclude on Day 2. In such cases, the teacher should include a wrap-up activity at the end of each class meeting (perhaps an Exit Ticket designed as a progress check-in) and start-up activity at the beginning of each class meeting (perhaps a quick write to reactivate learning from the previous day). These activities can help maintain continuity during a longer lesson sequence.

**Providing Access for All**

The Learning Plan section of the unit template includes a section labeled **Universal Design for Learning (UDL)**. UDL is the foundation to the overall curriculum planning process. This section represents a key consideration in curriculum development: making the content and skills instruction accessible to all students, whatever their learning styles, special needs, levels of English proficiency, school experiences, or degrees of engagement. Addressing this aspect of curriculum design is a major challenge for teachers but essential for student success.

The DYS philosophy and framework for providing access for all is based on **Universal Design for Learning (UDL)**, “a set of principles for curriculum development that give all individuals equal opportunities to learn” from the Center For Applied Special Technology (CAST). These principles, represented in the graphic organizer on p. 2.1.5, call for instruction that includes:

**Multiple means of engagement**, to tap into learners’ interests, offer appropriate challenges, and increase motivation. This principle involves the brain’s affective networks and concerns the “why” of learning: how learners get engaged and stay motivated; how they are challenged, excited, or interested. The goal is to stimulate interest and motivation for learning.
Multiple means of representation, to give diverse learners options for acquiring information and knowledge. This principle involves the brain’s recognition networks and concerns the “what” of learning: how we gather facts and categorize what we see, hear, and read. Identifying letters, words, and an author’s style are recognition tasks. The goal is to present information and content in different ways.

Multiple means of action and expression, to provide learners options for demonstrating what they know. This principle involves the brain’s strategic networks and concerns the “how” of learning: how we plan and perform tasks, how we organize and express our ideas. Writing an essay is a strategic task. The goal is to differentiate the ways that students can express what they know. (Adapted from CAST)

These UDL guidelines, developed by the Center for Applied Special Technology in Wakefield, Massachusetts, extend to education the architectural concept of universal design—the idea that buildings and landscapes should be constructed to accommodate a wide spectrum of users rather than retrofitted to address particular needs. Incorporating UDL principles in the classroom means planning for diversity rather than coping with it. Giving students a variety of options for receiving, processing,
and engaging with content helps ensure that they have equal opportunity for success. The CAST website offers a wealth of resources for creating access in units.

SEE: CAST  
www.cast.org

One of the most difficult aspects of implementing UDL is clarifying the aims of instruction. For example, if a unit Performance Task requires students to make a series of recommendations based on data and mathematical modeling, does the principle of providing “multiple means of action and expression” mean that a student can create a PowerPoint or give a speech instead of writing a paper? Probably. If the objectives in this unit include showing work in three ways (data table, equations, and graphs), those expectations certainly can be met in a visual or oral format. But preparing the recommendations by constructing several paragraphs of coherent prose would also be a legitimate objective, and in that case the options for action and expression could include how the prose gets written: with the aid of a graphic organizer or essay template, speech-to-text software, teacher conferencing and scribing, or other means of reaching the goal. The key is to focus on that goal and to remove as many barriers as possible.

The options for action and expression listed in the previous paragraph are examples of Differentiated Instruction (DI), a set of teaching practices that encourage teachers to “adjust the curriculum and presentation of information to learners rather than expecting students to modify themselves for the curriculum” (Hall et al.). Three elements of the curriculum may be differentiated:

1. Content (materials, tasks, instruction)  
2. Process (student grouping, classroom management strategies)  
3. Products (ongoing assessment, exploration, expectations for student responses)

DI’s theoretical framework is different from UDL’s, and it focuses more on accommodating individual needs than on building in accessibility, but many of its techniques are consistent with UDL principles. For example, the DI practice of giving students the option to work in pairs as they search for evidence in a text is consistent with the UDL teaching method of providing opportunities to practice with support (see other examples in Hall et al.).

Driven by changes already happening at the higher education levels and the need to prepare students for the 21st century workplace, Blended Learning provides teachers with a variety of ways to address student needs, differentiate instruction, and provide data for instructional decision-making. Blended Learning is the combination of digital content and activity with face-to-face content and activity. Some options available to DYS teachers to use as digital tools include Gizmos at ExploreLearning and NBCLearn, which can help introduce a topic or reinforce students’ understanding of it. To encourage a blended learning approach, DYS teachers may use a content management system such as Edmodo or PBWorks to host online learning activities that they have designed. These tools allow for independent learning as well as opportunities for students to share their work with the class community and receive feedback or have conversations with other students.

There is a blurred line between Technology Integration and Blended Learning. The video “Blended Learning and Technology Integration” can help teachers understand the difference. Technology Integration and Blended Learning are both great ways to get students engaged and motivated to learn.

SEE: Blended Learning and Technology Integration  
www.youtube.com/watch?v=KDGt7K9FfGsCKg

All students benefit from curriculum designed using UDL principles, implemented using DI practices, and taught in a Blended Learning environment, but these inclusive approaches are especially helpful for English Language Learners (ELLs), of whom there are many in DYS schools. To better serve ELLs, Massachusetts has joined the World-Class Instructional Design and Assessment Consortium (WIDA), which promulgates English Language Development Standards.
(aligned with the Common Core), Performance Definitions for English proficiency levels (see chart on p. 2.1.7), Model Performance Indicators for particular standards and levels, formal assessments, and much more. One essential element of the WIDA initiative is its focus on academic language:

Everything we do at WIDA revolves around the significance of academic language and how to empower language learners to reach for success (WIDA, Academic Language).

Another is its “Can Do Philosophy,” which “embraces inclusion and equity” and focuses on expanding students’ control of academic language, as illustrated in the performance definitions chart.

The WIDA website provides many helpful resources for teachers.

SEE: WIDA
www.wida.us

These include a search page (WIDA, Search) that allows the user to enter a grade level cluster, framework (formative or summative), language domain (listening, speaking,

Performance Definitions for the Levels of English Language Proficiency in Grades K-12

At the given level of English language proficiency, English language learners will process, understand, produce, or use:

<table>
<thead>
<tr>
<th>Level</th>
<th>Description</th>
</tr>
</thead>
</table>
| 6 Reaching | • specialized or technical language reflective of the content areas at grade level  
• a variety of sentence lengths of varying linguistic complexity in extended oral or written discourse as required by the specified grade level  
• oral or written communication in English comparable to English-proficient peers |
| 5 Bridging | • specialized or technical language of the content areas  
• a variety of sentence lengths of varying linguistic complexity in extended oral or written discourse, including stories, essays, or reports  
• oral or written language approaching comparability to that of English-proficient peers when presented with grade-level material |
| 4 Expanding | • specific and some technical language of the content areas  
• a variety of sentence lengths of varying linguistic complexity in oral discourse or multiple, related sentences, or paragraphs  
• oral or written language with minimal phonological, syntactic, or semantic errors that do not impede the overall meaning of the communication when presented with oral or written connected discourse with sensory, graphic, or interactive support |
| 3 Developing | • general and some specific language of the content areas  
• expanded sentences in oral interaction or written paragraphs  
• oral or written language with phonological, syntactic, or semantic errors that may impede the communication, but retain much of its meaning, when presented with oral or written, narrative, or expository descriptions with sensory, graphic, or interactive support |
| 2 Beginning | • general language related to the content areas  
• phrases or short sentences  
• oral or written language with phonological, syntactic, or semantic errors that often impede the meaning of the communication when presented with one- to multiple-step commands, directions, questions, or a series of statements with sensory, graphic, or interactive support |
| 1 Entering | • pictorial or graphic representation of the language of the content areas  
• words, phrases, or chunks of language when presented with one-step commands, directions, WH-, choice, or yes/no questions, or statements with sensory, graphic, or interactive support  
• oral language with phonological, syntactic, or semantic errors that often impede meaning when presented with basic oral commands, direct questions, or simple statements with sensory, graphic, or interactive support |

reading, writing) and other search criteria, and receive a set of model performance indicators for various levels of English proficiency. For example, a search using the criteria 9-12, summative, writing, and claim yields an example on critical commentary. The performance indicators range from “Reproduce critical statements on various topics, illustrated models, or outlines” for Level 1 to “Provide critical commentary on a wide range of issues commensurate with proficient peers” for Level 5. This kind of detail can help teachers set challenging but realistic expectations for ELLs in their classrooms.

Accessibility as a Lens

Sorting out all the details of complex initiatives such as Universal Design for Learning (UDL), Differentiated Instruction (DI), and WIDA can be daunting, so it is helpful to focus on what they have in common: an understanding that curriculum planning is not just about what is taught but also about who is learning.

The main goal of all three programs is equity—making instruction accessible and relevant to all students—and all are compatible with the Understanding by Design (UbD) method of unit design. But rather than using UDL, DI, and WIDA as checklists for evaluating the accessibility of a Learning Plan, it is better to use accessibility as a lens for viewing assessments and lessons as they are being created. That way, UDL, DI, and WIDA principles can be infused throughout the unit. During implementation, the teacher can feel secure in knowing that tools and scaffolds and options are already in place—and concentrate on monitoring progress.

Works Cited


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WIDA. Search the ELP Standards. wida.us/standards/ELP_StandardLookup.aspx.

Acronym Recap

DI Differentiated Instruction
UbD Understanding by Design
UDL Universal Design for Learning
WIDA World-Class Instructional Design and Assessment Consortium
Teaching Mathematics to Student Mathematicians ......................................................... 3.1.1
  The Student as Mathematician ................................................................................. 3.1.1
  Engaging Students and Teaching Concepts ............................................................ 3.1.3
  Building Mathematical Knowledge and Skills ....................................................... 3.1.5
  Assessment in Mathematics ..................................................................................... 3.1.7
  Teaching Mathematics in DYS ................................................................................. 3.1.9
The Student as Mathematician

_Education is not the filling of a pail, but the lighting of a fire._
—Yeats

Scenario 1
Students are sitting in rows that are facing a whiteboard. The teacher tells students that they are going to find the volume of various objects today, and they are going to begin with cubes. The teacher draws a cube on the board and labels each side as measuring 3 inches. She tells the students that the equation to find the volume of a cube is the length of a side cubed or \( V = a^3 \). Students write this equation in their notebooks. She asks them what the volume of this cube is. Students answer 27 cubic inches. Next, the teacher draws a rectangular prism on the board with side measurements of 2, 3, and 4 inches. The teacher tells students that the equation for solving volume of a rectangular prism is \( V = w \cdot l \cdot h \). Students write this equation in their notebooks, too. The teacher asks them what the volume of this prism is, and the students answer 24 cubic inches. This continues until the teacher has given students the equations for multiple prisms. Then, the teacher gives students a worksheet that asks them to find the volume of various prisms. At the end of the class, students turn in their worksheets. The teacher will review the worksheets to make sure that students are using the correct volume formulas to solve the problems on the worksheet. She will review the formulas again tomorrow if students did not perform well on this Formative Assessment.

Scenario 2
Students are sitting at tables of four, facing each other to allow for communication and collaboration. The teacher gives students various objects with different volumes (a cube, a cylinder, a rectangular prism) and asks them which object can hold more water. Students discuss their thoughts with each other and develop an educated guess. The teacher asks groups to share what they discussed before giving each group water to measure and pour into their containers to test their hypotheses. Students might be surprised at what they find, and they talk to their groups about why they think some containers hold more water than they originally thought. Then, the teacher explains to students that we can’t always “fill” a container to find out how much it holds. We sometimes need an equation that we can use that will tell us what the volume of a prism is. Instead of telling students the equations that they should use, the teacher gives students rulers and three numbers—each representing the volume of one of the prisms. Based on what they learned from the water activity, students need to match each prism to its cubic volume and discover the formulas for finding the volume of each prism. They must work together and discuss why they want to attempt to find the volume formulas in a certain way. The teacher gives small 1-inch cubes for students to use as manipulatives as they try to find the formulas. By the end of class, all students may not have discovered all three volume formulas, but they should be able to explain how they attempted to solve the problem, what went wrong, and how they attempted to fix it. The teacher asks students to justify the formulas that they came up
with, and students explain how they used the cubes to help them do that. The students who did find the correct formulas present their justifications to the class so that all students understand why the formulas work and must be true. All students reflect on their process and write down their thoughts on an Exit Ticket before they leave class.

**Emphasizing Problem-Solving Skills**

Hopefully, we realize that we want our students sitting in the second classroom where students see themselves as mathematicians and not simply as students who are solving problems on a worksheet. Our memories of learning mathematics might be closer to the first scenario and might include memorizing formulas, plugging numbers into equations, and attempting to find the correct answer, but we want our 21st century classrooms to be more engaging and more focused on the way that we approach solving problems, not just on solving problems and finding the “right” answer. Teachers should emphasize the process, should encourage students to rethink their approach, and should encourage students to think about how they would approach a similar problem differently if they were given it again. After all, students are going to be exposed to mathematical problems throughout their lives. They might not intuitively know the exact formula or way to approach solving a problem, but we want them to have the confidence that they can figure it out if they apply what they do know to the situation and use their problem-solving skills.

Students will apply their mathematical skills and problem-solving skills in daily situations such as:

- Determining if it is more cost effective to buy something “in bulk” versus buying a smaller size container that is on sale
- Determining whether or not the statistics we see are purposely misleading us
- Determining the quickest route to get somewhere
- Determining how much profit can be made based on sales after paying any upfront costs
- Determining whether a fuel-efficient car is more cost-efficient over the lifetime of the car, considering how many miles a person drives

“Our memories of learning mathematics might be closer to the first scenario and might include memorizing formulas, plugging numbers into equations, and attempting to find the correct answer, but we want our 21st century classrooms to be more engaging and more focused on the way that we approach solving problems, not just on solving problems and finding the ‘right’ answer.”

- Determining how to read data that we are given
- Determining how much paint we need to purchase to paint a room in our home

These problem-solving situations do not even take into account the math that students might need to use in future careers that they might have, such as engineers, computer programmers, carpenters, or insurance agents. Even if students do not go into careers that utilize mathematical skills on a daily basis, it is apparent that students will encounter mathematical problems that they need to solve in their daily activities. We need to give them the confidence, skills, and knowledge that they need to solve the problems that they will encounter.

Most importantly, perhaps, is the idea that giving students the confidence to solve mathematical problems will give them confidence to solve all problems that they encounter in their lives. When students are faced with any complex problem, there is always something to try, always a way to learn from what they try, and always a benefit to trying something else. We have to cultivate that belief in students because most students don’t come into our
classrooms believing that. In our mathematics classrooms, we teach students to notice patterns and draw inferences, which are critical skills that they will need in any career that they go into. Giving students the confidence that they can use higher-order thinking skills and showing them that they are capable of solving complex problems are the most important things we can teach them. If we can teach them to use this growth mindset, we will serve them well in life.

**Engaging Students and Teaching Concepts**

If we want our students to care about the skills and knowledge that we are teaching them in our mathematics classroom, we need to make sure that we are engaging them in issues that they care about and make sure that we are guiding them through the process of learning difficult concepts. In order to do this, there are key ideas that we should keep in mind.

**Cognitive Dissonance**

Cognitive Dissonance is defined as the confusion that results from having conflicting information from different sources about a subject (“Cognitive Dissonance”). This can be an uncomfortable feeling for students in the math classroom who are struggling to find the solution to a complicated problem, but this is something that students need to struggle with—it helps them grow and learn. Often, teachers become uncomfortable when their students are uncomfortable, and teachers want to jump in to help students solve the problems that they are struggling with. After all, we decided to become teachers because we wanted to teach students and help them learn. Teachers have to become comfortable with the idea that allowing students to be uncomfortable and struggle through a complex problem is helping them learn. When we interrupt students’ thought processes by jumping in with the way to solve a problem, we are stifling their critical thinking and engendering a sense of helplessness that we might not be able to undo.

The best thing that teachers can do to help students who are struggling is allow them to fumble and stumble. When students are struggling, the teacher can ask students to verbalize their thinking by saying things such as “Tell me what you’re thinking” or “Show me what you tried.” The teacher should encourage students to think out loud in an attempt to work out where they began to stumble so that they can begin to find another approach. After a student shows the teacher that he or she attempted to solve a problem, the teacher should prompt the student by asking:

What happened when you tried that? What do you think you will try next? Why do you think this new approach will work better?”

Instead of telling students how to re-approach a problem, we need to guide their thinking to allow them to find the solution.

When students are feeling frustrated, the teacher should emphasize to students that experimentation is a good
thing, even if it leads to failure at times. Students learn from failure just as much as they learn from successes. In fact, Stanford Professor of Mathematics Jo Boaler tells us that “when we make a mistake, synapses fire. (A synapse is an electrical signal that moves between parts of the brain when learning occurs.)” (Boaler). She tells us that students do not even need to be aware that they made a mistake in order for their brains to grow from it; they simply need to make a mistake. Teachers need to emphasize to students that making mistakes is okay. Boaler says, “Children and adults everywhere often feel terrible when they make a mistake in math. They think it means that they are not a math person” (Boaler). We want to teach students that there is no such thing as a “math person” or a “non-math person.” All students are capable of grappling with complex ideas.

Even though we want students to struggle with complex ideas, it is also important for teachers to address the idea of misconceptions explicitly with students. A misconception isn’t simply a mistake like a computational error. It is a fully-formed way of looking at something that may have taken root in our students’ thinking years earlier, which is why it is so hard to dislodge from our students’ minds. Teachers need to give students a concrete experience that shows that the misconception doesn’t work. The teacher must be aware that it is possible that even after this concrete experience, students may still revert back to their misconception.

A common misconception students have is that they confuse volume with surface area. If a teacher asks a student what the volume of a cube is that is $5 \times 5 \times 5$, the student may have the misconception that volume is the same as surface area and say that it is 150. Working with plastic cubes will give students a concrete experience to show them that surface area and volume aren’t the same, but the teacher also needs to tell students explicitly that they have a misconception. Teachers should not be afraid to use the word and say to students, “I think you have a misconception here.” This allows students and teachers to have a common language when discussing areas of concern. The teacher can then use formative assessments that address the misconception.

“\[\text{We want to teach students that there is no such thing as a ‘math person’ or a ‘non-math person.’}\]”

**Slow Math**

Another important idea for us to address with our students is that we shouldn’t rush through problems in an attempt to solve them as quickly as we can. We need to slow down. Students have a lot of binary beliefs about math: they either know a concept or they don’t; they “get” math or they don’t. Encouraging students to slow down, think about the problem they are given, and experiment with possible solutions is one way to combat these binary beliefs.

We want our students’ mindsets about math to change from this binary belief to an effort-based belief where students believe that if they slow down and don’t give up, they can solve any problem given to them. In order to do this, we should talk explicitly to our students about effort-based beliefs and get students to talk openly about their binary beliefs. Once students address the binary beliefs that they hold about themselves and about learning math, we can help our students confront them and rid themselves of those beliefs.

Many students may hold binary beliefs because they don’t invest much effort in trying to understand math. Teachers should encourage students by telling them that effort matters—if they work on something enough, they will build understanding and will overcome the obstacles that they imagine are in their way. Teachers should work on building a classroom environment that places strong value on students being able to explain why something does or doesn’t make sense. Praise students when they stick with a problem and don’t give up. Even if students do not come to the correct answer, praise students for working toward a solution. This praise will go a long way toward encouraging students to stick with a problem and not rush toward a solution.
When students are working slowly through mathematical problems, the teacher should be careful about subtle cues that will influence students’ choices. A lot of students get through math class by learning to read the teacher’s face. Teachers need to be careful not to change their facial expressions when students are approaching a problem incorrectly because this will cue students into the fact that they need to change their thinking. This robs students of the learning that comes from working problems out for themselves. We should also keep in mind that we don’t want to discourage students from solving a problem in their own way. What might appear incorrect to us might be an approach to solving the problem that we didn’t think of. If we stop students from thinking about the way that they are approaching a problem and we impose our own way of thinking on them, we are encouraging them to rush through the experience of solving problems instead of encouraging them to slow down and learn from their process.

Building Mathematical Concepts

Out of all the subjects that students take in school, math might be the domain where you are most likely to find people who are trying to fill a pail rather than light a fire. People often think about math as being delivered on high from an authority, probably because that was the way that they were taught when they were in school. We need to remember that many of the people who chose to become math teachers are the ones who were successful in this type of classroom environment. This, however, is not the way that most students are able to learn. Teachers need to be aware that the vast majority of students are not faring well under traditional instruction. Teachers should change their way of thinking and begin thinking about math in terms of a grassroots approach where learners work together to build their mathematical understanding themselves.

Building mathematical vocabulary is essential to building mathematical concepts. To help students with the numerous new vocabulary words that they will be exposed to in each unit, the teacher should establish concepts with students first, then put a name to it. For example, a teacher may be used to saying to the class, “Today we will learn about volume. Let’s start by defining the word volume. Volume is the amount of space that an object occupies.”

Instead, the teacher can bring in two bottles and ask students, “Which bottle do you think will hold more water?” The students can debate this question and use informal and intuitive approaches to find an answer. Here, students are spending a lot of time thinking about volume without being given the word “volume” or a definition of it. At the end of the discussion, the teacher can say, “Now we see that bottle A holds more water. Mathematically, we would say that bottle A has a greater volume than bottle B. The volume of a container tells us how much it can hold.” Even though this may feel counter-intuitive for a lot of people, this will allow students to attach meaning to a vocabulary word instead of asking students to learn a word without being able to attach meaning to it.

Building Mathematical Knowledge and Skills

KNOWING, ASKING, DOING, THINKING: Using an Inquiry Approach

Before engaging students in any new unit of study, teachers must have a good understanding of the prior
knowledge that students are bringing with them so that teachers can remediate as necessary before engaging students in new material. Teachers also want students to have a strong sense of the skills that they already have so that students feel confident when approaching new tasks.

To begin the inquiry process, teachers should encourage students to ask questions about what they see and encourage them to wonder how they can solve problems that they encounter. Teachers should encourage this curiosity in students and should allow students to bring a wide variety of questions into the classroom that relate to the unit of study. To truly engage in inquiry, students should then attempt to solve problems using whatever tools and prior knowledge they have at their disposal.

Teachers can guide students and offer suggestions for tools that students can use; however, teachers want to allow students to struggle and think since this is where real problem-solving techniques are born. Even though students might not find the solution to the problem the first or second time that they attempt to solve it, students should reflect on the approach that they took, think about why the approach didn’t work, and try again. We want our students to reflect on the process that they engaged in so that they can learn from what they did and apply that learning to new situations.

**READING:**
**Not Just Words on a Page**

The Common Core State Standards ask that all teachers teach students to read their content area materials. Reading in the mathematics classroom can be especially difficult for students who are otherwise strong readers because “research has shown that mathematics texts contain more concepts per sentence and paragraph than any other type of text. They are written in a very compact style; each sentence contains a lot of information with little redundancy. The text can contain words as well as numeric and non-numeric symbols to decode” (ASCD). Therefore, it is important for teachers to teach students to slow down, read carefully, and use reading strategies to approach difficult text.

Whenever possible, teachers should bring additional readings into the classroom to connect the concepts being studied to real-life situations. For example, when students are studying statistics, teachers should bring in articles that can be analyzed for misleading statistics. Students can analyze how statisticians came up with the statistics that are given and could point out why the statistics painted an accurate or inaccurate picture of the truth.

It is also important for teachers to remember that in the mathematics classroom, reading does not mean simply reading articles about the mathematical concepts students are studying. The narrow definition of reading should be widened to include teaching students to read charts, data, graphs, and other content area concepts.

**WRITING:**
**Writing to Learn**

Students should be engaging often in writing activities in the math classroom. Teachers should think of writing in the math classroom not simply as writing mathematical equations and labeling axes on graphs, but as writing to explore ideas, thinking, and concepts. Students often do not know what they know until they write it down, so we should allow students to use writing as a tool to figure out what they know. This should be taking place daily through formative assessments that ask students to use writing to learn strategies.

As much as possible, teachers should have students write as part of their Performance Tasks. This writing should be as authentic as possible in regards to the task that students are asked to do. If teachers are creating their Performance Tasks in a GRASPS format, they should consider having
the end product include some type of authentic writing. The more students write in their math class, the more they will be comfortable expressing their mathematical thinking through writing.

**SPEAKING AND LISTENING:**
*A Place of True Collaboration*

The mathematics classroom should be a place of true collaboration. Students should be able to verbalize their thought processes to their peers and their peers should listen and respond to what they hear. We need to teach our students that by sharing our thought processes, we are helping others learn from our way of thinking. Often, there is more than one way to solve a problem, but if we don’t share our ideas with our peers and listen carefully when they speak, we will not learn those other ways of approaching problems.

Speaking and listening skills should be utilized and practiced daily through discussions of Do Now activities, working on solutions to problems, and in summing up what was learned and explored. Students will also practice their speaking and listening skills during presentations of their Performance Tasks. Teachers should ensure that classmates are listening carefully to their peers’ presentations so that they can give feedback that is constructive and will push their thinking further.

**LANGUAGE:**
*Using the Vocabulary of Mathematics*

Giving students the language that mathematicians use will help them explain their reasoning in terms that everyone will understand. Teachers want to introduce vocabulary words to students in a manner that will allow them to visualize what the concept is, work with the concept, and internalize the new words. Once students become familiar with new vocabulary, they will be able to use these new words and communicate with their peers in a common language.

Students can easily become overwhelmed by the number of new words that they encounter in every unit of study in their mathematics classroom. Teachers should provide students with word walls for them to reference to ensure that students are using proper vocabulary when discussing mathematical ideas. Students should also create their own vocabulary notes, such as T-Charts, so that they have reference guides of their own.

**NUMERACY:**
*Reasoning with Numbers*

The basis of numeracy skills asks students to be able to add, subtract, multiply, and divide numbers in order to solve problems. As students progress in their learning, this foundation allows students to explore more complex ideas and work with numbers in more complicated ways. Being numerate in the secondary mathematics classroom means that students are able to analyze data, apply mathematical concepts, and solve a variety of questions. We want our students to leave our classrooms feeling confident that they can work with numbers to solve problems.

Students will work with numbers in a variety of ways in the high school math classroom. They will analyze data sets, solve equations, work with real and imaginary numbers, and use arguments to convince others that their line of reasoning is valid. These arguments can take many forms, such as diagrams, equations, graphs, or analogies.

**Assessment in Mathematics**

**Performance Tasks**

Teachers should strive to assess students as authentically as possible in their classrooms. This means that the typical paper and pencil tests that we may have taken when we were students aren’t the only types of assessments that we want to use with our students today. In order to truly assess whether or not students understand concepts, students should be given authentic assessments that
If students start to believe in the idea of writing to learn, then they will be more willing to start writing...

Formative Assessments

Formative Assessments should be used daily, both to help students solidify their thinking and to give teachers an understanding of the concepts and skills that need to be revisited in subsequent classes. Do Nows and Exit Tickets are quick, easy ways for teachers to check in with students informally, while still providing the teacher with the knowledge that they need to inform their teaching. Teachers may also choose to use online formative assessment tools such as Kahoot or Socrative. Students enjoy using these online tools to practice their skills and teachers can collect data quickly and efficiently through these games.

Testing

While we want to avoid using typical paper and pencil tests as the only means of assessment in the mathematics classroom, there could be times when the teacher finds that these types of tests are appropriate. One appropriate use of this type of test would be as a Pre-Assessment to quickly assess gaps in students' knowledge. These tests could include multiple choice, short answer, and open response questions that ask students to apply their understanding of concepts to new problems. When used sparingly, these types of tests and quizzes can be used to familiarize students for the types of standardized tests that they will have to take.

MCAS: Preparing Students for High-Stakes Tests

We need to stop thinking that we need to prepare students for MCAS testing by giving them dozens of practice exam questions to solve. If our math class is a place of informed instruction and a place where students are solving complex problems and writing about how they solved them, they will be well prepared to take the

“If students start to believe in the idea of writing to learn, then they will be more willing to start writing...”
MCAS exam. If all we do to prepare our students to take the MCAS is give them practice tests, students will not feel prepared when they see problems on the test that are unfamiliar. They will not have the problem-solving skills to approach questions that are difficult. Teaching students to reason their way through problems and to try a new approach when their first approach doesn’t work are much more effective skills to teach in our classrooms.

Here’s an example from the 10th grade MCAS test to illustrate how students can solve a problem without needing to know the algebraic equation to solve it:

Nancy, Bryan, and Jamie combined their money to purchase a laptop. Together they paid a total of $490 for the laptop, including tax.

Nancy paid $50 more than Bryan paid. Bryan paid twice as much as Jamie paid. How much did Nancy pay?

A. $108
B. $176
C. $226
D. $295

If students do not know how to set up the algebraic equation to solve this problem, they can still reason their way through it. The first thing that they should notice is that Nancy paid a lot more than her friends. She paid more than Bryan, and Bryan paid more than Jamie. If they had just split the cost three ways, each would have paid about $163, so we can eliminate the first two answers right away. Now, we can check the last two possible answers. We’ll start with $295 because it seems like Nancy paid a lot more than her friends, and that is the largest number. If Nancy paid $295 and she paid $50 more than Bryan. Bryan paid $245. We see that $295 and $245 is over $490, so that can’t be the right answer. We can then try $226. If Nancy paid $226, Bryan would have paid $176, and Jamie would have paid half that, $88. Those add up to $490, so C is the correct answer.

Teachers should also keep in mind that students should be writing about math during the entire year. Writing shouldn’t be something they do just to get ready for MCAS. If students start to believe in the idea of writing to learn, then they will be more willing to start writing on an open-response question even if they don’t know the answer. They will learn that sometimes, the act of writing actually changes their understanding of a problem.

Teaching Mathematics in DYS

Teaching math to students in the DYS context can pose many challenges for teachers. Students come into the DYS classroom with varying levels of knowledge and skills, and the teacher has to address the gaps that some students have in order to teach the standards that are set forth in the CCSS. To fill these gaps, teachers need to have clear goals in mind for each lesson. While planning for lessons, teachers must think about the prerequisites that are needed to be successful and then find a quick way to assess for these skills. If students are lacking the prerequisite knowledge, teachers should address these gaps with students. This can be done through small group instruction and through the use of websites that allow students to practice their skills.

Explore Learning Gizmos is one website that teachers might find helpful because it provides simulations that students can watch and also allows students to practice their skills on the computer.

SEE: www.explorelarning.com

Likewise, teachers might find Utah State University’s National Library of Virtual Manipulatives (NVLM) website helpful because it allows students to work with virtual manipulative material to practice their skills.

SEE: nlvm.usu.edu

The IXL website is another source for practice activities to help students build their skills.

SEE: www.ixl.com

Another difficulty in the DYS context is that, within one classroom, students could be studying different contents and be in various grades. To personalize instruction for these students, the teacher might consider presenting a “mini-lesson” to all students that relates to a topic that all students can explore in their math content area. Students can then break into groups at various stations throughout
the classroom to get more individualized instruction from the teacher and to work through problems with their peers. Teachers can utilize technology such as Edgenuity to help with this.

One of the struggles that teachers may notice in the classroom is that many students lack a “growth mindset.” They believe that they can’t understand the concepts that are being taught and therefore don’t try. To help students with this, the teacher should meet students in their zone of proximal development, meaning that the work needs to challenge students, but can’t be so challenging that students want to give up before they even begin. Teachers should celebrate effort, even when students get the wrong answer because this helps students find the right answer. Teachers can also use online tools that give students quick feedback to encourage them and help them develop their skills.

Connecting the math that students are learning to real-life examples is essential to keep students interested in what they are learning. Wherever possible, and especially in Performance Tasks, teachers should show students how the math that they are learning applies to their lives and will be used in future careers. Teachers may consider inviting professionals into the classroom, where appropriate, to talk to students about how they use math in their fields.

The transient population of the DYS setting also poses a challenge for teachers. Throughout this guide, adaptations have been provided to guide teachers through adapting units for shorter or longer-term settings. Teachers can think of the discrete skills that are being taught in each unit and think of the larger unit as being composed of “mini-units.” Formative Assessments in the larger units can serve as Summative Assessments in shorter-term settings.

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**Works Cited**


Massachusetts Department of Elementary and Secondary Education. “Release of Spring 2016 Test Items (with Answer Key).” www.doe.mass.edu/mcas/2016/release
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Introduction to Algebra 1

Why is algebra important to learn? Just look around. Its usefulness can be seen everywhere—if you take the time and look.

Introduction

Why do we need to learn algebra? Professor Zalman Usiskin, former Director of the University of Chicago School Mathematics Project passionately answers this question in his article “Why Is Algebra Important to Learn? (Teachers, this one’s for your students!).”

Most people realize that they need to know arithmetic. Whole numbers, fractions, decimals, and percentages are everywhere. ... Just pick up a newspaper or magazine, open to any page at random, and count the numbers on it. You may be surprised at how many there are.

Algebra seems different. Scan the same newspaper and you are not likely to see any algebraic formulas. The need to know algebra isn’t as obvious. ... In actuality, however, its usefulness is all around us. But for those of us who don’t know where or how to look, it is often hidden. It is well worth the effort to dig a little deeper to uncover the many ways this discipline is at work in the world and why mastering it greatly enriches our lives.

In a nutshell:

*Algebra provides a very simple language for what you are doing.*

—Usiskin

Algebra is the mathematical language we use to clearly and precisely describe things (entities), patterns, and processes. Algebraic formulas exist in every walk of life. Some of those formulas are handy, for example: converting Fahrenheit temperatures to Celsius and vice versa, determining area when installing flooring or calculating earned-run averages. Even income tax, sales tax, and car loans, almost every money matter involves some formula. Algebra also helps you to describe the relationships between variables. What happens to your health as you age? How is energy use affected by population? How much can you spend and still stay on your budget?

As students study algebra, they will begin by building on familiar arithmetic concepts. They will learn to discover mathematical patterns and begin to develop strategies for justifying a pattern. Additionally, they will learn that algebra is the branch of mathematics that uses letters in place of some unknown numbers. As they gain important mathematical vocabulary and discover how to use algebraic operations, equations, inequalities, functions, exponents, quadratics, and polynomials, they will learn how to solve these real-world problems. In the process, students will learn to use deductive reasoning to determine if they agree with the solution to a problem and test their conclusions to determine if they are valid—this is the basis of mathematical thinking.

Another way to engage students in the study of algebra is to identify the many career pathways that involve algebraic knowledge and skills. Career paths that utilize the skills and knowledge learned in Algebra 1 include:

- Animators who use linear algebra to create the movements of characters or objects in animated films or video games.
Market research analysts who gather statistical data to determine consumer buying habits.

Engineers and architects who use exponents to help them design and build machinery, structures and equipment.

Bankers, financial consultants, and accountants who use equations to determine monthly mortgage and auto loan payments, and give investors information about how much they are earning on savings accounts, pension plans and stock portfolios.

EMT/nurses who calculate proper dosages of medical drugs, IV drip rates, drug titration (such as insulin), as well as body mass or glycemic indexes (Beerman).

Many students may assume that highly-skilled scientists who work in “rocket propulsion … would require much more advanced math than algebra. It is true that more advanced math is necessary to understand every facet of [this] and other advanced topics.” This advanced math will be expressed in algebra, even if some of the symbols represent properties and functions that Algebra 1 students would not yet have learned. “However, many of the fundamental principles can be understood using only the tools in [introductory] algebra. For example, the equations that describe how a spacecraft orbits the earth only involve algebra” (Gibson).

There are multitudes of practical ways to use algebra in everyday life. Potential careers that rely on the knowledge of algebra abound in the fields of business, economics, construction, medicine, computer graphics, technology, and more. Connecting students with these very real, useful, and empowering applications of algebra can keep them interested even when learning how to manipulate equations.

Algebra 1 Course Content

The three key shifts in the new Common Core State Standards for Mathematics ask teachers to provide greater focus on fewer topics, coherence—linking topics and thinking across grades—and rigor in their classrooms. The third key shift, rigor, is especially important. This rigor does not refer to making math more challenging for students or introducing high-level skills at an earlier age. Instead, it asks teachers to implement what the CCSS calls three elements of rigor: conceptual understanding, procedural skills and fluency, and application (“Key Shifts”).

While all three elements of rigor are addressed in the following exemplar units, teachers should pay careful attention to the “application” element, which asks “students to use math in situations that require mathematical knowledge. Correctly applying mathematical knowledge depends on students having a solid conceptual understanding and procedural fluency” (“Key Shifts”).

As much as possible, teachers will ask students to apply their new understandings to new situations and real-life tasks. The Summative Assessments in each unit ask students to engage in rigorous tasks where students will apply their newfound knowledge to a new problem they might encounter in their daily lives. As the CCSS states, “Students must be able to access concepts from a number of perspectives in order to see math as more than a set of mnemonics or discrete procedures” (“Key Shifts”).

By applying their learning to new contexts, they will ensure mastery of the knowledge and skills addressed in the CCSS.

The Algebra 1 course addresses several Essential Questions that will help students focus on the larger understandings in mathematics:

- What sorts of real-world situations and processes are linear, reciprocal, exponential, quadratic, or radical?
- How can we use mathematical modeling to make better decisions?
- Where in the real world do we see examples of reciprocal patterns, positive or negative linear patterns, exponential growth or decay, quadratic relationships?
- How does learning about linear, reciprocal, exponential, quadratic, and radical functions help me understand real-world problems?
- Why is it important to collect numerical measurements of entities or processes as the basis of algebraic data analysis?
How can people use statistics to misrepresent patterns in data that are neither valid explanations nor predictors of the real world?

While many of these questions are best suited to be discussed within a certain season or unit, most can be discussed throughout the entire year and in other disciplines. Just as teachers of English teach students to create convincing arguments using reasoning and logic, teachers of math will instruct students to create convincing arguments using numerical data to support their claims expressed precisely in algebraic language.

Embedded in all three Algebra 1 exemplar units are the Common Core State Standards for Mathematical Practice.

The Three Seasons of Algebra 1

**Linear Functions and Equations** focuses students on the foundations of algebra and prepares them for future math courses. Students begin by learning algebraic expressions and the basic algebraic properties that guide computation and manipulation.

This first season of the year includes units on the language of algebra, solving linear equations, linear functions and inequalities, and linear modeling.

**Linear Functions and Equations, Exponential Functions, and Radical Functions** focuses on bringing a basic understanding of linear equations, graphs and functions and a foundational understanding of exponents and powers. Students will graph and find the solution for systems of linear functions, use exponents to develop exponential growth and decay models, extend the rules of exponents to rational exponents, and learn about functions.

The second season includes units on systems of linear functions and inequalities, properties of exponents, exponential functions, exponential modeling, and simplifying radical expressions.

**Quadratics and Polynomial Functions, and Statistics** focuses on functions and how to use data to develop various mathematical models. They will continue working with various types of quadratic and polynomial functions and learn how x-intercepts are used to solve quadratic and polynomial equations. Additionally, students will learn to analyze and interpret data on a single count or measurement variable.

The third season includes units on simplifying polynomial expressions, factoring quadratic and polynomial expressions, solving quadratic equations, quadratic modeling, and fundamentals of statistics.

Teaching Algebra 1 in DYS Schools

As students begin the study of Algebra 1 in the “Language of Algebra” unit or finishing the year with their study of “The Fundamentals of Statistics,” teachers need to ensure that students can connect what they are learning in the classroom to their everyday lives. Whenever possible, teachers should provide students with real-life examples of the mathematical skills that they are learning in the classroom so that students see a purpose in what they are learning. Teachers can provide these examples through readings, job connections, or through Summative Assessments that ask students to solve real-world problems. Throughout the sample units, examples have been provided for teachers to encourage this discussion.

Using a variety of instructional methods will encourage student participation. Teachers will provide students with as many hands-on activities as possible so that students can use manipulatives to engage in the skills they are learning. Teachers will also provide students with videos, PowerPoint, games, and readings to encourage learning through a variety of media. Additionally, teachers will want to use technology as much as possible to give students the skills that they will need to be successful in future careers. Many useful websites have been
provided for teachers to engage students in the content and skills of this course.

The study of Algebra 1 can prove challenging for many students because they do not understand how algebraic symbols connect with the real world. The number of equations and vocabulary words that students encounter for the first time can be daunting. Teachers will create anchor charts and will have students create personalized reference sheets to aid them while they acquire this knowledge. While it is important that students learn many of these equations and new vocabulary terms, the emphasis should not be on simply memorizing the words and equations, but rather on using the equations to solve problems and using the newly acquired vocabulary to articulate what they are learning and how they are solving problems.

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Usiskin, Zalman. “Why is Algebra Important to Learn? (Teachers, this one’s for your students!)” American Federation of Teachers. Spring 1995.
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Scheduling Options:

Algebra 1 can be structured in two ways to accommodate the various abilities and needs in the DYS classroom. It can be taught either as an AT LEVEL (12 months) or FOUNDATIONAL LEVEL (24 months) class. The topics listed (as units) are in pedagogical order, but are not the only units that can be taught in each season. Teachers may develop other units to cover the standards in each of these seasons. The exemplar units included in this Guide are indicated with an asterisk (*).
Reading the Algebra 1 Scope and Sequence Chart

The amount of information contained in the Scope and Sequence on the following pages may seem overwhelming at first. The best way to study it is to read across from left to right. The keys below on this page offer guidance on how to properly access the Scope and Sequence Chart on pp. 4.2.2 to 4.2.3.

The Scope and Sequence is COLOR-CODED. Each color is important, and its meaning and the main highlights in DYS pedagogy are described in the key in the LEFT column below. The RIGHT column shows the Algebra 1 topics and the seasons when they are taught during the academic year.

Topics listed with an asterisk (*) in the Scope and Sequence have exemplar units in this Guide.

<table>
<thead>
<tr>
<th>Scope and Sequence Chart Key</th>
<th>Algebra 1 Topics</th>
<th>Seasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>The GOLD header rows identify columns for Topics, Emphasized Standards, Essential Questions, Transfer Goals, and Performance Assessments for each of the seasons in mathematics.</td>
<td>Linear Functions and Equations</td>
<td>FALL</td>
</tr>
<tr>
<td>The GREEN row contains the FALL season.</td>
<td>Linear Functions and Equations, Exponential Functions, and Radical Functions</td>
<td>WINTER</td>
</tr>
<tr>
<td>The BLUE row contains the WINTER season.</td>
<td>Quadratics and Polynomial Functions, and Statistics</td>
<td>SPRING</td>
</tr>
<tr>
<td>The RED row contains the SPRING season.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>The GRAY row across the bottom contains the eight Common Core State Standards for Mathematical Practice (SMP).</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Mathematics | Algebra 1, Chapter 4

### SCOPE AND SEQUENCE

<table>
<thead>
<tr>
<th>Topics</th>
<th>Emphasized Standards</th>
<th>Essential Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Linear Functions and Equations</strong>&lt;br&gt;1. The Language of Algebra&lt;br&gt;2. Solving Linear Equations&lt;br&gt;3. Linear Functions&lt;br&gt;4. Linear Inequalities&lt;br&gt;5. Linear Modeling*</td>
<td><strong>Number and Quantity:</strong> Reason quantitatively and use units to solve problems (N-Q 1, 2, 3).&lt;br&gt;<strong>Algebra:</strong> Interpret the structure of linear, quadratic, and exponential expressions with integer exponents (A-SSE 1a). Create equations that describe numbers or relationships (A-CED 1, 2, 3, 4). Solve equations and inequalities in one variable (A-REI 1, 3). Represent equations and inequalities in two variables (A-REI 10).&lt;br&gt;<strong>Functions:</strong> Interpret linear, quadratic, and exponential functions with integer exponents that arise in applications in terms of the context (F-IF 4, 5, 6). Analyze functions using different representations (F-IF 7a, 9). Construct linear models and solve problems involving linear models. (F-LE 1a, 1b, 2, 5).&lt;br&gt;<strong>Statistics:</strong> Summarize, represent, and interpret data (S-ID 6a, b, c) Interpret linear models (S-ID 7, 8, 9).</td>
<td>How can we use mathematical (linear) modeling to make better decisions? What sorts of situations are linear? How do patterns help me understand equations in mathematics?</td>
</tr>
<tr>
<td><strong>Linear Functions and Equations</strong>&lt;br&gt;6. Systems of Linear Equations and Inequalities</td>
<td><strong>Number and Quantity:</strong> Extend the properties of exponents to rational exponents (N-RN 1, 2).&lt;br&gt;<strong>Algebra:</strong> Create equations that describe numbers or relationships (A-CED 1, 2, 3). Solve equations and inequalities in one variable (A-REI 1, 3). Solve systems of equations (A-REI 5, 6). Represent and solve equations and inequalities graphically (A-REI 10, 11, 12).&lt;br&gt;<strong>Functions:</strong> Interpret functions that arise in applications in terms of the context (F-IF 4, 5). Analyze functions using different representations (F-IF 7a, 7e, 8b). Construct and compare linear and exponential models (F-LE 1a, b, c, 2, 3, 5).&lt;br&gt;<strong>Statistics:</strong> Summarize, represent, and interpret data (S-ID 6a).</td>
<td>Where in the real world do I see examples of exponential growth and decay? How does learning about exponential functions help me understand real-world problems?</td>
</tr>
<tr>
<td><strong>Exponential Functions</strong>&lt;br&gt;1. Properties of Exponents&lt;br&gt;2. Exponential Functions&lt;br&gt;3. Exponential Modeling*</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Radical Functions</strong>&lt;br&gt;1. Simplifying Radical Expressions</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Quadratics and Polynomial Functions</strong>&lt;br&gt;1. Simplifying Polynomial Expressions&lt;br&gt;2. Factoring Quadratic and Polynomial Expressions&lt;br&gt;3. Solving Quadratic Equations&lt;br&gt;4. Quadratic Modeling</td>
<td><strong>Algebra:</strong> Interpret the structure of linear, quadratic, and exponential expressions with integer exponents (A-SSE 2). Write expressions in equivalent forms to solve problems (A-SSE 3a, b). Perform arithmetic operations on polynomials (A-APR 1). Create equations that describe numbers or relationships (A-CED 1, 2). Solve equations and inequalities in one variable (A-REI 4a, b). Solve systems of equations (A-REI 7). Represent and solve equations and inequalities graphically (A-REI 10, 11, 12).&lt;br&gt;<strong>Functions:</strong> Interpret linear, quadratic, and exponential functions with integer exponents that arise in applications in terms of the context (F-IF 4, 5). Analyze functions using different representations (F-IF 7a, 7b, 8a, 9). Build new functions from existing functions (F-BF 3). Construct and compare linear, quadratic, and exponential models and solve problems (F-LE 3).&lt;br&gt;<strong>Statistics:</strong> Summarize, represent, and interpret data on a single count or measurement variable. Use calculators, spreadsheets, and other technology as appropriate. (S-ID 1, 2, 3). Summarize, represent, and interpret data (S-ID 6a).</td>
<td>How can people use statistics to misrepresent data? Why is it important to collect and analyze data?</td>
</tr>
<tr>
<td><strong>Statistics</strong>&lt;br&gt;1. Fundamentals of Statistics—Data on a Single Variable*</td>
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</table>

### Common Core State Standards for Mathematical Practice (SMP):<br>1. Make sense of problems and persevere in solving them.<br>2. Reason abstractly and quantitatively.<br>3. Construct viable arguments and critique the reasoning of others.<br>4. Model with mathematics.
### Transfer Goals

- **Application of Learning**
  - Students will represent linear data in multiple ways, e.g. graph, equation, scatter plot.
  - Students will draw logical conclusions from linear data.
  - Students will be able to represent a real-life situation that involves a constant rate as a linear equation and graph and then make meaningful predictions based on the data.
  - Students will show how multiple representations of linear data are related and then apply this to many other types of mathematical data in the future.

- **Performance Assessment**
  - **To Buy or Not to Buy a Car?**
    Students will apply their understanding of linear modeling to compare all of the costs involved in owning a car: one that is already owned vs. one that is being considered for purchase. Students will start by brainstorming (and identifying) which car-related costs are one-time (up front) and therefore will comprise the y-intercept and which car-related costs are yearly (which will comprise the slope). Then, students will use linear modeling to create a presentation/poster/storyboard of their findings and develop a detailed recommendation based on those costs (using the vocabulary of linear modeling, independent variable, dependent variable, y-intercept, slope, coefficient, variables, constants, interpretation, etc.) about whether or not buying the newer car is a good economic decision. The students will also write an introduction and a three-paragraph summary and conclusion. Ideally, the owner of the car as well as other students in the classroom will be the audience for the presentations.

- **Sustainability Project**
  Students will spend two days researching, analyzing data, and preparing a written and visual presentation for the following assignment:
  Imagine that your class has received a request from the non-profit organization, Learning for a Sustainable Future (LSF). As a part of their classroom outreach, LSF is asking students to research one topic related to environmental sustainability and prepare a report that explains their findings and makes predictions or draws conclusions about environmental sustainability. Each student should research a topic of their choosing, find an exponential data set, then create multiple representations of the data, analyze the data, and prepare a presentation that discusses the implications of the data with regard to sustainability. After the students receive peer feedback on their presentations, they will make adjustments before presenting to the entire class and passing in their final report.

- **Reaction Time Project**
  Students will create data, represent it in various ways, compute the measures of center, and interpret the data’s shape and variance. They will use their knowledge of statistics to help them make an informed decision about how to present their data to their peers and teacher.
  Using the “Reaction Timer” website students will create two sets of data, one a baseline of their reaction time and one a measure of their reaction time with an option (distractor) that students have chosen.
  The final project will culminate in a presentation, during which students will present both sets of data and interpret them individually and in comparison to each other.

### Common Core State Standards for Mathematical Practice (continued):

5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.
The Fall Season prepares students to understand the world of algebra. They are introduced to key skills and understandings that will lay the foundation for the rest of the year and future math courses. The Fall Season prepares them to apply their skills to more complicated problems as the year progresses.

The unit outlined here, “Linear Modeling,” is the fifth unit in the Fall Season of Linear Functions and Equations and will take about three weeks to complete. It presumes prior knowledge of:

- measurement in conventional and metric units
- the form of a linear equation: \( y = mx + b \)
- representing data in tables, graphs, or equations
- the meaning of linearity: constant rate of change (i.e., slope) from one data point to the next
- constructing an \( x-y \) coordinate grid with properties and units labeled on the axes

Students lacking these prerequisite skills should spend additional time reviewing before beginning the unit.

The unit asks students to think about the following Essential Questions:

- How can we use mathematical (linear) modeling to make better decisions?
- What sorts of situations are linear?

Students will research, write, and read about real-world linear situations and how the situations have an impact on their lives. They will also learn to collect, chart, and graph linear data, and solve a variety of problems alone, with partners, or with the whole class. Students will explore questions throughout the unit by looking at real-world examples of linear relationships and complete a series of linear modeling tasks that will culminate in a research-based performance assessment.

In the Performance Task students will use linear modeling to compare the costs and values of operating a currently owned, older car with those of a newer car that might be purchased. Students must first collect and represent linear data in three formats which they then interpret to draw conclusions. Students will present their findings orally and with the aid of PowerPoint, poster, or storyboard. They end with cost-based recommendations to the car owner and other audience members.
This unit and its Essential Questions focus on standards:

**A-REI.10:** Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). Show that any point on the graph of an equation in two variables is a solution to the equation.

**F-LE.1a:** Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

**F-LE.1b:** Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

**F-LE.2:** Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

**F-LE.5:** Interpret the parameters in a linear or exponential function (of the form \( f(x) = bx + k \)) in terms of a context.

**S-ID.6a:** Fit a linear function to the data and use the fitted functions to solve problems in the context of the data. Use given functions fitted to data or choose a function suggested by the context (emphasize linear and exponential models).

**S-ID.6c:** Fit a linear function for a scatter plot that suggests a linear association.

**S-ID.7:** Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

After studying this unit, students will be able to recognize linear situations, collect and interpret data that shows a constant rate of change, and solve real-world problems using linear modeling.

After completing this unit, teachers may choose to discuss with students how linear modeling is used in the workplace. Teachers could invite a small business owner, an EMT or nurse, or a mechanic into the classroom to discuss practical applications to future careers.

For adaptation ideas for this unit, see p. 4.3.3 on the right
## Linear Functions and Equations: Linear Modeling

Adapting This Long-Term Unit for Short-Term Programs

### Plan 1 (Long)

**FALL SEASON—Linear Modeling: Long-Term Programs**

<table>
<thead>
<tr>
<th>MONDAY</th>
<th>TUESDAY</th>
<th>WEDNESDAY</th>
<th>THURSDAY</th>
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<tr>
<td><strong>Week 1</strong></td>
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<td>Lesson 1:</td>
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<tr>
<td>Windows can be</td>
<td>Tying the Knot</td>
<td>Still Tying the</td>
<td>At the Fair</td>
<td>At the Big E</td>
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<td>a real PANE!</td>
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<td><strong>Week 2</strong></td>
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<td>Lesson 5:</td>
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<td>Lesson 6:</td>
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<td>Lesson 7:</td>
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<td>At the Big E</td>
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<td>Important Rates</td>
<td></td>
<td>Graphs and Rates</td>
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<td>for Car Ownership</td>
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<tr>
<td><strong>Week 3</strong></td>
<td>Lesson 8:</td>
<td>Lesson 9:</td>
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<tr>
<td>Getting to Know</td>
<td>To Buy or Not to</td>
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<tr>
<td>the Kelley Blue</td>
<td>Buy a Car?</td>
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<td>Book</td>
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### Plan 2 (Short)

**FALL SEASON—Linear Modeling: Short-Term Programs**

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<tr>
<th>MONDAY</th>
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<td><strong>Week 1</strong></td>
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<td>Lesson 1:</td>
<td>Lesson 2:</td>
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<td>Lesson 4:</td>
<td>Lesson 5:</td>
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<tr>
<td>Windows can be</td>
<td>Tying the Knot</td>
<td>Still Tying the</td>
<td>At the Fair</td>
<td>At the Big E</td>
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<tr>
<td>a real PANE!</td>
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<td>Knot</td>
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</tr>
<tr>
<td><strong>Week 2</strong></td>
<td>Lesson 5:</td>
<td>Lesson 6:</td>
<td>Lesson 7:</td>
<td>Lessons 8 and 9:</td>
</tr>
<tr>
<td>At the Big E</td>
<td></td>
<td>Important Rates</td>
<td>Graphs and Rates</td>
<td>Getting to Know the</td>
</tr>
<tr>
<td>(continued)</td>
<td></td>
<td>for Car Ownership</td>
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<td>Kelley Blue Book</td>
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<td>To Buy or Not to</td>
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<td></td>
<td></td>
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<td></td>
<td>Buy a Car?</td>
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</tbody>
</table>

For short-term program settings (Plan 2), Lesson 5 (At the Big E) could be shortened by providing students with the information on the costs of materials and rentals for their booth, rather than allowing students to conduct their own research.

The Performance Task (To Buy or Not to Buy a Car?) can also be designed as a shorter assessment if the introduction to the Kelley Blue Book website is abbreviated and the research is completed for the students. The teacher can choose 5 or so very different cars: Prius, pickup truck, mini-van, Subaru, sports car. The teacher can find Kelley Blue Book information—such as mpg, trade-in values, and price—and put these details in a chart. Students can then use a spinner to determine which car is already owned and which car is being considered. A spinner can also be used to determine how many miles are driven each month and the amount of maintenance the owned car needs per month. At this point the students will be able to complete their equations, data table, and graphs for the project (pp. 6 and 7 of the To Buy or Not to Buy a Car? Project Packet) and then present a more condensed version of their findings and recommendations.
UNIT PLAN

For Long-Term Programs

Linear Modeling
Designed by: J. Stevens and J. Baer-Leighton
Theme or Content Area: Algebra I—Linear Functions and Equations
Duration: 8 Lessons / 3 Weeks

Emphasized Standards (High School Level)

ALGEBRA

A-REI.10: Understand that the graph of an equation in two variables (equations include linear, absolute value, exponential) is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). Show that any point on the graph of an equation in two variables is a solution to the equation.

FUNCTIONS

F-LE.1a: Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

F-LE.1b: Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

F-LE.2: Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

F-LE.5: Interpret the parameters in a linear or exponential function (of the form $f(x) = bx + k$) in terms of a context.

STATISTICS AND PROBABILITY

S-ID.6a: Fit a linear function to the data and use the fitted functions to solve problems in the context of the data. Use given functions fitted to data or choose a function suggested by the context (emphasize linear and exponential models).

S-ID.6c: Fit a linear function for a scatter plot that suggests a linear association.

S-ID.7: Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

For Empower Your Future Connections, see p. 4.5.1
Essential Questions *(Open-ended questions that lead to deeper thinking and understanding)*

How can we use mathematical (linear) modeling to make better decisions?
What sorts of situations are linear?

Transfer Goals *(How will students apply their learning to other content and contexts?)*

Students will represent linear data in multiple ways, e.g. graph, equation, scatter plot.
Students will draw logical conclusions from linear data.
Students will be able to represent a real-life situation that involves a constant rate as a linear equation and graph and then make meaningful predictions based on the data.
## Learning and Language Objectives

*By the end of the unit:*

<table>
<thead>
<tr>
<th>Students should know...</th>
<th>understand...</th>
<th>and be able to...</th>
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</thead>
<tbody>
<tr>
<td><strong>y-intercept</strong></td>
<td>The point where the graph of a linear relationship crosses the y-axis is the value of the dependent (responding) variable when the independent (manipulated) variable is set to zero, a common starting point for many real-world investigations.</td>
<td>Locate the y- and x-intercepts on a linear graph. Identify the real-world significance of the x- and y-intercepts for a particular linear relationship.</td>
</tr>
<tr>
<td><strong>x-intercept</strong></td>
<td>The x-intercept for a linear relationship is the value of the independent (manipulated) variable when the dependent (responsive) variable is zero—few real-world, linear situations will have x-intercepts that have useful meaning. It answers the question, “What value do I have to set for the manipulated variable in order for the responding outcome to be zero?”</td>
<td>Use the linear equation to calculate the values for the x- and y-intercepts by substituting a value of zero for the other.</td>
</tr>
<tr>
<td><strong>Slope</strong></td>
<td>The slope of a line (rise over run) is understood in terms of the property units on each axis and represents the rate at which the responding variable changes when the manipulated variable is changed.</td>
<td>Calculate the slope of a linear graph (either direct or inverse) as the rise over the run on any triangle built off the trend line and calculated as the changes in the property values on each axis spanned by the two sides of the triangle. Express the slope of a linear graph with units that are a ratio of the two divided properties (e.g., 12 miles per hour, $1.50 per pound, 8 cm per knot)</td>
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</table>
### Constant rate

The orientation of the constant slope of a linear graph indicates whether the relationship between the variables is direct or inverse.

The value of the constant slope of a linear graph indicates by what factor the dependent (responsive) variable changes when the independent (manipulated) variable is changed.

### Variables and constants

Variables are traits, properties, or characteristics (quantities) whose numerical values can change.

A constant is a trait, property, or characteristic (quantity) that does not change value during the particular situation that is being modeled.

Because the rate at which two variables change relative to each other is constant throughout most real-world situations, the slope of their graph will be constant.

In a real-world situation, the rate at which one variable changes relative to the other is translated into the slope value of a linear equation, while the most obvious beginning or zero condition is translated into either the $x$- or $y$-intercept.

<table>
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<tr>
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<tr>
<td><strong>Constant rate</strong></td>
<td>The orientation of the constant slope of a linear graph indicates whether the relationship between the variables is direct or inverse. The value of the constant slope of a linear graph indicates by what factor the dependent (responsive) variable changes when the independent (manipulated) variable is changed.</td>
<td>Distinguish between linear and non-linear data in a data chart, a scatterplot of data points, or the trend line of a graph. Describe in words how two variables that are directly and that are inversely related change relative to each other. Sketch the slopes of linear graphs that illustrate direct and inverse relationships. Identify from the slope of a linear graph whether the relationship between the variables is direct or inverse.</td>
</tr>
<tr>
<td><strong>Variables and constants</strong></td>
<td>Variables are traits, properties, or characteristics (quantities) whose numerical values can change. A constant is a trait, property, or characteristic (quantity) that does not change value during the particular situation that is being modeled. Because the rate at which two variables change relative to each other is constant throughout most real-world situations, the slope of their graph will be constant.</td>
<td>Explain the meaning of the values for the slope and $y$-intercept in a linear equation or graph representing a real-world situation in terms of the two variables being graphed.</td>
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</table>
### Students should know...

**Solving for a variable**

- Linear equations are used to calculate the value for one variable that corresponds with the value for the other.
- Linear equations are used to make predictions and draw conclusions about how the value of one variable would respond to changes in the other.

### understand...

- Given a real-world problem with a linear relationship, identify the unknown variable whose value is being sought.
- Rearrange an equation for a given, unknown variable by isolating it alone on one side of the equal sign.
- Interpret a linear equation to describe the rate at which one variable changes relative to the other.
- Calculate the value for each variable when the other is zero.

### and be able to...

- Assign values, either negative or positive, for either the dependent and independent variables and the constants in a linear equation.
- Use the order of operations to evaluate an equation to find the value of the unknown variable isolated on one side of the equal sign.

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*The acronym PEMDAS stands for Parentheses, Exponents, Multiplication, Division, Addition, Subtraction.*
Massachusetts DYS Education Initiative—Mathematics—2017 Edition | Chapter 4, Section 4

### Desired Results

<table>
<thead>
<tr>
<th>Students should know...</th>
<th>understand...</th>
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<tbody>
<tr>
<td>Linear modeling of real-world situations</td>
<td>Because any real-world situation is characterized by unique patterns in how its variables change, linear graphs and equations derived from measuring these variable changes will have unique slope and y-intercept values. Complex situations often involve more than one change pattern occurring simultaneously between its variables, each pattern producing its own linear graph or equation which can be analyzed together to answer complex questions.</td>
<td>Create linear equations for real-world situations that include a change in one variable compared to a second that is either directly or inversely related. Use multiple graphs on the same axes to represent multiple, simultaneous change patterns in a real-world situation to answer complex questions.</td>
</tr>
</tbody>
</table>
Assessment Evidence

Quality questions raised and tasks designed to meet the needs of all learners

Performance Task and Summative Assessment (see pp. 4.4.32-4.4.34)

Aligning with Massachusetts standards

To Buy or Not to Buy a Car?

Students will apply their understanding of linear modeling to compare all of the costs involved in owning a car: one that is already owned vs. one that is being considered for purchase. Students will start by brainstorming (and identifying) which car-related costs are one-time (up front) and therefore will comprise the $y$-intercept and which car-related costs are yearly (which will comprise the slope). Then, students will use linear modeling to create a presentation/poster/storyboard of their findings and develop a detailed recommendation based on those costs (using the vocabulary of linear modeling, including independent variable, dependent variable, $y$-intercept, slope, coefficient, variables, constants, interpretation) about whether or not buying the newer car is a good economic decision. The students will also write an introduction and a three-paragraph summary and conclusion. Ideally, the owner of the car as well as other students in the classroom will be the audience for the presentations.

Pre-Assessments (see pp. 4.4.15-4.4.28)

Discovering student prior knowledge and experience

Lesson 1: Do Now: Students sketch a large, rectangular, party-size pizza and determine how many corner slices, edge slices, and middle slices there would be if it was an 8 by 6 slice pizza. Students will determine which relationships are linear.

Lessons 2/3: Do Now: Students are given a set of equations and a set of situations and are asked to match the situation to the equation that it represents.

Lesson 4: Do Now: Students will list all the possible types of booths that may appear at a fair.


Lesson 6: Do Now: Inquiry questions about costs related to owning a car.

Formative Assessments (see pp. 4.4.17-4.4.34)

Monitoring student progress through the unit

Daily review and assessment activities.

Lesson 2: Students will share and display their equations with the class. After the students display their equations, the teacher will ask the students a series of questions. The answers to these questions will inform both teacher and students of their level of understanding regarding the $y$-intercept, slope, and the linear relationship between the knots and the length of the yarn.
Lesson 4: Students will describe linear situations based on equations. This activity will inform teachers and students about their understanding that $y$-intercepts can be positive or negative and allow students to create real-world examples that explain the meaning of the equations provided.

Lesson 7: Students will play “telephone graph-making” to assess students’ ability to make graphs that express given rates and conversely, read a graph and determine the rate. This activity will help the teacher determine how skillfully students can convert rates into graph form and vice versa.

Lesson 8: Students will sort car costs into “one-time costs” or “yearly costs,” which will help the teacher determine if the students understand the difference between the one-time costs associated with buying a vehicle and the recurring costs.

Lesson 9: (DAY 2) Students will review their equations and those of their classmates. They can make estimations about which cars are “worth it” and which aren’t. This activity will help students formulate recommendations based on the data collected and inform the teacher if the student recommendations are accurately based on the data.
Multiple Means of Engagement

This is the *why* of learning. It is what makes students engage or disengage. Throughout the unit plan, the student will be provided with as many choices in the level of challenge and complexity as possible in order to recruit and sustain engagement. For example, the teacher will encourage and support students in setting their own personal, academic, and behavioral goals. The teacher will use many strategies to guide students, including reminders, guides, rubrics, checklists, and prompts among other things that focus students on self-regulatory goals. Student tasks will be varied, allowing for active participation, exploration, and experimentation. The teacher will provide differentiated models, scaffolds, and feedback, as well as content material that is culturally relevant and responsive to student’s backgrounds. Most important is that teachers design assignments and tasks with authentic outcomes, and that are purposeful and convey meaning to real audiences.

The lessons in this unit are designed so students can use the discovery process to learn mathematical skills and concepts. Hands-on activities such as “Tying the Knot” provide students with physical examples of linear modeling. The students also apply these concepts and skills to real-world situations, e.g., What happens mathematically when knots of equivalent size are tied in a string, rope, etc.? What is qualitatively and quantitatively important when creating a highly competitive fair booth? The Summative Assessment at the end of the unit is an authentic Performance Task with an audience and purpose, e.g., determining economies of scale when deciding whether to buy a newer car. Students are also provided with a range of choices to fit their interests (e.g., creating and solving their own word problems, creating real-world situations to explain equations, etc.) and the level of challenge; extensions are provided for many lessons. Additionally, students have access to high-tech tools, such as graphing calculators, Gizmos, or online graphing calculators, or low-tech tools such as graph paper, chart paper, post-it notes, and math manipulatives. The color, design, and layout of graphics, and sequencing and timing may also be adjusted to meet student needs. Evaluative emphasis should be placed on process, effort, and improvement. Formative Assessments are designed to invite personal response, self-evaluation, and reflection. Student choice is also provided when completing internet research so students are engaged in summative and performance assessments. Students should be given problems and be shown math examples that address a wide range of diversity and learning profiles in the classroom. Where possible, assignments and brainstorming should be done in pairs and/or small heterogeneous cooperative learning groups.

Multiple Means of Representation

This is the *what* of learning. There are many pathways to conveying information to students. Throughout the unit, the teacher will provide information and materials in several modalities such as diagrams, vocabulary cards, and word walls, posters, and charts with formulas for calculations; and models, videos, and audio for text. The teacher will also demonstrate concepts through hands-on activities.
The way information is displayed should vary, including size of text, images, graphs, tables or other visual content. Students are presented with information in multiple modalities and from a variety of sources (e.g., websites included in unit plan, Discovering Algebra text, teacher-generated materials, multi-media resources, and visual graphics/representations). Where possible, written transcripts for videos and auditory content should be provided. Text-to-speech software may be used when assignment packets require a significant amount of reading or important portions of the text can be highlighted. Information should be chunked into smaller elements, and complexity of questions can be adjusted based on prior knowledge competency. Reference sheets for examples, anchor charts, notes, vocabulary, and definitions can be differentiated for content, process, or product by differing the style of provided notes or providing pre-made flash cards or pre-formatted electronic versions for vocabulary and types of equations.

By presenting information through a variety of culturally responsive means focusing on how and why to manipulate equations, rates of change, lines of best fit, and different forms of equations, the unit builds understanding in students that algebra has real-world applications and relevant meaning. Accommodations intended to adjust the unit’s learning and language objectives, Transfer Goals, level of performance and/or content will be necessary for students with mandated, specially designed instruction described in their Individualized Education Programs (IEPs).

Multiple Means of Action and Expression

This is the how of learning. In learning activities students will be provided options for demonstrating what they know and can do. Students will have access to assistive technology and use multiple media. For example, students will have access to word processors with grammar checks, word prediction, and spell checkers. Students could complete projects by making PowerPoint presentations, videos, storyboards, or drawing illustrations. In addition, students will have access to calculators. The teacher will scaffold writing or composing activities using tools such as concept maps, outlining tools, or graphic organizers. Students may need sentence starters and story webs to complete writing or composing tasks. The teacher will also break down long-term goals into short-term reachable goals.

Performance Tasks can be differentiated by content, process, or product to address various learner profiles. Pre-collected data can be provided, for example the “Big E” activity, to scaffold student learning. Students have the opportunity to create short presentations using Internet research, Glogster, or PowerPoint to express their final recommendation for the “To Buy or Not to Buy a Car?” project. Students should be given high- and low-tech options to compose in multiple media, such as text, speech, drawing, illustration, comics, or storyboards. Students can use graphic organizers, such as KTL webs, concept mapping with Inspiration, or drawings by hand, checklists, sticky notes, and mnemonic strategies, to better understand and demonstrate comprehension of the material. Opportunities for collaboration and whole-class discussion should be provided as needed.

Accommodations intended to enhance learning abilities, provide access to the general curriculum, and provide opportunities to demonstrate knowledge and skills on all performance tasks will be necessary for students with applicable Individualized Education Programs (IEPs) and could benefit all learners.

For Empower Your Future Connections, see p. 4.5.1
Literacy and Numeracy Across Content Areas

For each of the standards and each content progression, students will be required to communicate orally or in writing, their interpretation of the data and the conclusions they draw. Every level will require reflection for understanding, error analysis, and open discussion to fully understand material.

Reading

Students will be presented with information, including step-by-step directions and word problems, and will need to read and understand content in order to fully participate in the lesson. Selected readings, word problems related to algebra, and websites will help students gain a more complete understanding of the topics covered in the lessons.

Writing

Students will have a number of written assignments. Throughout the unit, daily “Do Nows” and word problems ask students to explain their reasoning and prove their answers through writing, while end-of-lesson written discussion questions, self-assessments, Exit Tickets, or feedback forms provide Formative Assessments. Real-world applications of linear modeling require students to write introductions and extended conclusions in order to present their findings. For the Performance Task, students will write a formal presentation that includes comparative research, cited evidence, and recommendations based on data.

Speaking and Listening

Students will learn the processes for engaging in a range of academic conversations using multiple perspectives and effective questioning through small group discussions, and formal and informal presentations. Many of the Hooks and Do Nows, as well as the Review and Assessments, require discussion and students are asked to share predictions and outcomes.

Language

Students will use content-specific Tier 2 and Tier 3 vocabulary to explain the mathematical concepts discussed in this unit. Examples of Tier 2 words include: evaluate, solve, calculate, concept, linear, variable, and inverse. Examples of Tier 3 (domain-specific words) include: y-intercept, x-intercept, slope, and constant rate. Vocabulary anchor charts will also help students remember important vocabulary and concepts related to linear modeling.

Numeracy

Opportunities for learning algebra are found in the use of statistics, graphs, formulas, tables, and time lines. Extrapolation, inference, and interpretation are all skills that are based in numeracy.

Students will develop skills in making sense of problems and persevering to solve them. They will learn to reason abstractly and construct viable arguments, as well as critique the reasoning of other people. It is important that they learn to model various real-world problems with mathematics. They will develop skills in learning how to use appropriate tools and understand how to attend to precision. Students will also look for and make use of structure.
Resources (in order of appearance by type)

Print


Websites

Lesson 1

Lesson 2

Lesson 8

Materials (most located in the Supplement)

Chart paper, markers, metric and U.S. rulers, magnifying glass

Lesson 1: Anchor Chart for Mathematical Concepts in Linear Modeling Activity Sheet p. 4.6.1
Lesson 1: Windows Can Be a Real PANE! Activity Sheet pp. 4.6.2-4.6.5
Lesson 2: Measurement Pre-Assessment Activity Sheet pp. 4.6.6-4.6.7
Lesson 2: Tying the Knot (When Did You Learn to Tie Your Shoes?) Activity Sheet pp. 4.6.8-4.6.10
Lesson 2: \(y = mx + b\) (Lab Extension) Activity Sheet pp. 4.6.11-4.6.15
Lesson 3: Still Tying the Knot (Ever Been to a Wedding?) Activity Sheet pp. 4.6.16-4.6.17
Lesson 4: At the Fair—A Line is Worth a Thousand Words Activity Sheet pp. 4.6.18-4.6.21
Lesson 5: Graphing Skills Pre-Assessment Activity Sheet p. 4.6.22
Lesson 5: My Booth Will Have a Longer Line than Your Booth Project Packet pp. 4.6.23-4.6.32
Lesson 6: Telephone Graph-Making Activity Sheet p. 4.6.33
Lesson 8: Getting to Know the *Kelley Blue Book* Activity Sheet pp. 4.6.34-4.6.37
Lesson 8: Car Talk (Interview Activity) Activity Sheet p. 4.6.38
Lesson 8: Car Intro: Pre-Assessment PowerPoint http://bit.ly/2qp5RVb
Lesson 9: To Buy or Not to Buy a Car? Project Packet pp. 4.6.39-4.6.48
Lesson 9: Exit Ticket and Feedback Form Activity Sheet p. 4.6.49
PREREQUISITES: Math skills needed for this unit

Linear Modeling is the fifth and final unit in the Fall Season of Algebra 1. The sixth and final unit in *Linear Functions and Linear Equations* is located in the Winter Season of Algebra 1. The prerequisite math skills summarized below are taught in the four units that precede Linear Modeling (see Scope and Sequence chart). The following skills will be needed for students to successfully complete this unit.

Students should know:

- **How to measure any distance**
  - Using both the conventional and metric scales on a ruler
  - In inches and fractions of an inch, or in centimeters and millimeters

- **Slope-intercept form of a linear equation**: \( y = mx + b \) (or \( y = ax + b \))

- **Y-intercept**:
  - Where the line crosses the y axis; in a linear equation, \( b \) is the y-intercept
  - At the \( b \) value, \( x = 0 \)
  - In a real-world situation, the \( b \) value can be thought of as the “starting point” or “starting amount”

- **Slope basics**:
  - \( m \) is the slope and \( b \) is the y-intercept
  - \( m \) (slope) is calculated from two data points by using formula: \( \frac{y_2 - y_1}{x_2 - x_1} \)
  - Slope can be measured from the graph by calculating “rise over run”
  - Slope can be calculated from a table of data by calculating the change in \( y \) values divided by the change in \( x \) values
  - The slope is the “rate of change” of the line … a positive slope tilts “up” as the line moves from left to right; a negative slope tilts “down” as the line moves from left to right

- **Solutions to linear equations**:
  - There are infinite solutions to any linear equation
  - Each is an \( (x, y) \) pair that “work” in the equation
  - When graphed, the solutions form a straight line on the grid
  - The x-intercept is the \( x \) value when \( y = 0 \) (it is where the line will hit the x-axis)
  - The y-intercept is the \( y \) value when \( x = 0 \) (it is where the line will hit the y-axis)

- **How to represent linear relationships**:
  - Linear relationships can be represented as graphs, equations, data tables, or sets of data points
  - Identify graphs as linear or not
  - Identify sets of points as showing a linear relationship or not
  - Identify a data table as showing a linear relationship or not
  - Identify an equation as linear or not
  - Match graphs to equations to data tables that show the same linear relationship
Students should know (continued):

- **A linear relationship shows a constant rate of change from one data point to the next:** Relationships between two variables can show a predictable change from one data point to the next—but one that isn’t a constant change. (Inverse, exponential, quadratic relationships, for example, show predictable changes—but these are not linear)
  Relationships between two variables show no predictable change between one data point and the next.

- **Functions can be graphed using an x, y coordinate grid:**
  - Choose an appropriate scale to fit all of the data
  - The scale for the x- and y-axes do not need to be the same
  - Real-world problems require labels (units) on the axes

**Unit Notes**

The Linear Modeling Unit was designed with the goal of providing students with an in-depth series of real-world situations that require them to use linear modeling to make meaningful predictions based on data. The lessons begin with hands-on skill development that allows students to understand the concept of linearity both visually and mathematically using manipulatives and sketches. In later lessons, students are provided with more complex real-world problems (the kinds of problems that a teenager may encounter) and asked to find relevant data and use their knowledge of linear modeling to make predictions and recommendations. At first, students will work in groups or pairs to hone their mathematical reasoning skills and finally, working independently, will demonstrate how they can apply these skills in different contexts—the essence of the Transfer Goal.

**Activity Sheets** or **Project Packets** that accompany lessons are found in the Supplement. These packets carefully guide the students through the logical steps required for linear modeling. The teacher will notice that some activities begin with a humorous title on the topic and periodically, students will be asked to self-assess their comfort level using the mathematical skills. The activities and packets may seem “paper heavy,” but the explanations, instructions, and activities carefully reinforce the skills needed to work independently. The lessons themselves provide many non-traditional learning opportunities for students to work together and with the teacher to develop and test their skills. It should be emphasized that these linear modeling lessons have been field-tested in Algebra 1 classrooms for the last six years.
Outline of Lessons
Introductory, Instructional, and Culminating tasks and activities
to support achievement of learning objectives

INTRODUCTORY LESSONS
Stimulate interest, assess prior knowledge, connect to new information

Note: Teachers should vet all videos included in this unit according to program standards and
create templates or graphic organizers for students to monitor their comprehension of the material.

Lesson 1
Windows Can Be a Real PANE!

Goal
In this lesson, students will differentiate relationships between two variables that are linear and those that
are not linear. Students will recognize linearity both from a table and from a graph. The idea of a constant
rate (as seen on a straight line graph) will be the common feature that brings students from one real
application of linearity to the next throughout this unit.

Do Now (time: 5 minutes)
The teacher will instruct the class to imagine that there is a big, rectangular party size pizza in the middle
of the room. The pizza has corner slices, edge slices, and middle slices. The teacher will ask:

If it were an 8 by 6 slice pizza, how many of each type of slice would there be?

The teacher draws a picture, on the board or on paper, to show an 8 by 6 slice pizza.

Hook (time: 5 minutes)
The teacher will ask the students:

If we were to change the number of slices … let’s make it a 5 by 8 slice pizza instead…

How would the number of corner, edge, and middle slices change?

Again, the teacher will use a picture to help.

Presentation (time: 5 minutes)
The teacher will display these two questions for the students:

What does it mean for there to be a constant rate of change?

How can we recognize linearity from a set of data and from a graph?

The teacher should discuss these questions generally, helping students to understand the concepts of
constant rate, linearity, and set of data, and graph, without revealing the answers for the pizza example. The
concepts should be listed and defined on an anchor chart, which can be used for reference in future classes.
See the "Anchor Chart for Mathematical Concepts in Linear Modeling" on p. 4.6.1 in the Supplement.

Note: Linear and non-linear relationships are explained in the Review and Assessment portion of this lesson.
Practice and Application (time: 10 minutes)

The teacher will explain to the students that they are going to complete an activity—similar to the pizza slice problem—however, this activity involves window panes. The activity will lead students toward recognizing linearity as:

- A straight line graph, and
- As a constant rate of change from one data point to the next

As students work in pairs on the “Windows Can be a Real PANE!” activity, found on pp. 4.6.2-4.6.5 of the Supplement, the teacher should be sure that the students understand the three types of window panes and how they fit together to create square windows. Once the students can confidently understand the uses of the three types of panes (corner, edge, middle), they should be able to move smoothly through the activity.

The teacher should circulate among the groups to answer questions that the students may have.

The teacher should stop students periodically to compare answers.

Note: Numbers 3a and 3b on p. 4.6.3 are good check-in points.

The teacher will ask:

What combination of square windows will create an area of exactly 72 square feet?
For each window in the combination, how many of each type of pane will the builder need?

Then, allow the students to complete the remainder of the “Windows Can Be a Real PANE!” activity, completing the tables, explaining the patterns they observe, creating algebraic expressions, and graphing their data. If some students finish early, they can complete the CHALLENGE question on p. 4.6.5 of the activity.

Review and Assessment (time: 5 minutes)

Students should share their answers to Question 6 (p. 4.6.4) of the “Windows Can be a Real PANE!” Activity Sheet. The teacher will say,

Study your graph.
Which one(s) are linear? Which one grows the fastest?

The teacher can provide feedback to the pairs of students as needed. Next, the teacher should ask the students to refer back to the questions from the Presentation portion of the lesson. The students will be looking at the sketches of the pizza slices once again and the guiding questions:

The teacher will ask the students:

What kind of a relationship exists between the edge pieces and the side length?
A linear relationship, both are linear measurements.

What did you observe about the corner pieces?
The corner pieces are constant—always 4!

What did you discover about the relationship between the middle pieces and the side length?
They have a non-linear (quadratic) rate of change with side length.

The teacher should add the following description of linear and non-linear relationships to the anchor chart:

Edge Pieces: 0 \(\rightarrow\) 4 \(\rightarrow\) 8 \(\rightarrow\) 12 \(\rightarrow\) 16 \(\rightarrow\) 20 (rate of change = +4)

Middle Pieces: 0 \(\rightarrow\) 1 \(\rightarrow\) 4 \(\rightarrow\) 9 \(\rightarrow\) 16 \(\rightarrow\) 25 \(\rightarrow\) 36 (rate of change is +1 then +3 then +5 ...)
Lesson 2

Tying the Knot  *(What do slope and y-intercept mean in a real-world situation?)*

Lesson 2 is designed as an introduction to Lesson 3, “Still Tying the Knot.” The two lessons are intended to be taught over two consecutive days.

**Pre-Assessment**

In this lesson, students will need to measure lengths of yarn. It may be important for students to refresh their measuring skills with both a conventional U.S. ruler and a metric ruler. A measurement pre-assessment, “Measurement Using Conventional or Metric Rulers,” has been included on pp. 4.6.6-4.6.7 of the Supplement. This pre-assessment will allow students to practice using conventional U.S. and metric rulers to measure parts of a standard classroom pencil. Students will be able to discover where difficulties arise and fine tune their skills. The teacher will be able to determine if additional practice in measuring is needed before Lesson 2 begins.

**Goal**

This lesson will give students a hands-on example of a real-world situation that can be represented by linear equations (and graph and data table). Students will see how, because there is a constant rate of change, they can use their equation to make predictions. They will also see how the actual value of the slope in the equation is determined by the rate of change. Additionally, students will see that the heavier (wider) yarn will have greater rates of change—each knot uses up more yarn; while the thinner yarns will have smaller rates of change—each knot using up less yarn.

**Do Now** *(time: 5 minutes)*

Students will be given the set of equations and a set of situations below and are asked to match the situation to the equation that it represents. The teacher can record the equations and situations on separate index cards and have the students match them or can provide a handout with the information and have students note the correct pairings. Students should share their answers and explain how they came to their conclusions.

<table>
<thead>
<tr>
<th>Equations:</th>
<th>Situations:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y = 3x + 30)</td>
<td>Jamal owes his uncle $30.</td>
</tr>
<tr>
<td>(y = -3x + 30)</td>
<td>Each week he pays back his uncle $3.</td>
</tr>
<tr>
<td>(y = -30 + 3x)</td>
<td>There are 30 cookies in the jar.</td>
</tr>
<tr>
<td>(y = -30 – 3x)</td>
<td>Each minute, Lila takes three cookies from the jar.</td>
</tr>
<tr>
<td></td>
<td>It is negative 30 degrees at the top of Mount Washington at noon.</td>
</tr>
<tr>
<td></td>
<td>Each hour, the temperature drops three more degrees.</td>
</tr>
<tr>
<td></td>
<td>There are thirty students in class.</td>
</tr>
<tr>
<td></td>
<td>Each minute, three more students arrive.</td>
</tr>
</tbody>
</table>

**Hook** *(time: 5 minutes)*

The teacher will hold up a 30cm length of yarn and ask:

How many identical knots can be tied in this yarn?
Let the students write down their guesses. The teacher will ask the students to start tying knots carefully, being sure that each knot takes up approximately the same amount of yarn. The students should stop tying when no more knots can be made. Who was closest to the correct amount?

Note: Yarn longer than 30 cm. might not be permitted due to safety and security concerns. Teachers should check to see if certain materials and lengths are allowed.

Presentation (time: 5 minutes)
The teacher will provide the following question, and students will work with a partner to answer the question:

Is there any information you could have had beforehand that would have helped you make a more confident prediction?

Two of the answers that the students should consider are:
- Starting length of yarn (in metric units), and
- Length of yarn each knot takes up (in metric units)

The teacher should ask the students to turn and talk, then report back to the whole group with their answers.

Next, re-try the activity with a very different sort of yarn. (Much thicker or much thinner; longer or shorter...) The teacher will ask students to again make a prediction about the total number of knots that can be made.

Practice and Application (time: 30 minutes)
The teacher will give each pair of students a specific length and thickness of yarn. Each pair of students will also need a meter stick. Students will work through the “Tying the Knot” Activity Sheet found on pp. 4.6.8-4.6.10 in the Supplement.

Note: Teachers may want to cross-reference a condensed version of the “Tying the Knot” Activity, which can be found on pages 242 and 243 of Discovering Algebra (2002).

The teacher should circulate among the groups to answer questions that the students may have. The groups will likely complete the first page and part way through the second page. The teacher should check student progress and determine if most student pairs have completed or are close to working on question 11.

Review and Assessment (time: 10 minutes)
The teacher will provide the student pairs with chart paper and markers and ask the students to write down the following questions and display their equations on the chart paper. The teacher should consider preparing the chart paper with the questions ahead of time or posting the questions on the board. The teacher will then ask students to share their equations with the class.

Which yarn started the longest?
Students should choose the equation with the largest y-intercept.

Which yarn was the thickest?
Students should choose the equation with the largest slope, since thick yarn will use up more yarn per knot.

What is it about the relationship between the number of knots (x) and the length of yarn (y) that makes it a linear relationship?
Each knot uses the same amount of yarn.
Extension

Students can complete the “\(y=mx+b\)” lab activity using the Desmos Graphing Calculator to review the parts of a linear equation. See pp. 4.6.11-4.6.15 of the Supplement for the Activity Sheet.

Note: This activity may also serve as preparation for Lesson 3, “Still Tying the Knot.”

SEE: Desmos Graphing Calculator
www.desmos.com/calculator

INSTRUCTIONAL LESSONS

*Build upon background knowledge, make meaning of content, incorporate ongoing Formative Assessments*

Lesson 3

Still Tying the Knot

Lesson 3 is designed as a follow-up to introductory Lesson 2, “Tying the Knot.” The two lessons are intended to be taught over two consecutive days.

Goal

Students will see the connection between y-intercept and starting amount of string as well as the slope and rate of change. They will discover the connection by comparing their equations (and yarn) to those of their classmates. Students will also see the relationship between a negative slope and a rate of change that is decreasing. Finally, students will see how their equations (and a little bit of algebra) can be used to make predictions about number of knots and lengths of yarn.

Do Now (time: 5 minutes)

The teacher will show the students a new set of knotted yarns of various lengths and equations that represent each yarn. The teacher will ask the students to match the equations to the yarns.

Hook (time: 10 minutes)

The teacher will show students another yarn that has 5 knots in it and ask them:

Is there any way to find the starting length of yarn without untying any knots?

The teacher should prompt the students:

What kind of a plan do we need to answer the questions?

What should we take into consideration?

The teacher should give the students some time to come up with a plan that includes:

- Measuring the current length of yarn
- Tying one more knot
- Measuring new length of yarn
- Finding the difference between the two lengths (that is the amount of yarn used up by each knot)
- Multiplying that amount by 5 and then adding that amount to the first measurement of the yarn
Presentation (time: 10 minutes)
The teacher should explain that the students will be using these planning methods, their equations (and other equations) to make predictions. Today, student pairs will start with problem 12 in the “Tying the Knot” Activity Sheet from Lesson 2. The teacher should review problem 12 and the three sub-sections with the students before they begin work. This Activity Sheet can be found on p. 4.6.9 in the Supplement. At this point, the teacher should review the parts of the slope intercept form of a linear equation:
\[
y = mx + b \quad \text{(or} \quad y = ax + b)\]

Practice and Application (time: 15 minutes)
The students should finish the Lesson 2 “Tying the Knot” activity, problems 12-16 on pp. 4.6.9-4.6.10. As the students work, the teacher should circulate among the student pairs to answer questions.

Review and Assessment (time: 5 minutes)
The teacher will pass out copies of the “Still Tying the Knot” Activity Sheet, found on pp. 4.6.16-4.6.17 of the Supplement. The students will complete problems 1-3 in the Activity Sheet.

Extension
If time is available, students can complete problem 4 in the “Still Tying the Knot” Activity Sheet. See p. 4.6.17 in the Supplement.

Lesson 4
At the Fair

Goal
Students have seen a positive y-intercept—the starting length of yarn. They have seen a negative slope (the constant rate at which the yarn became shorter). What sort of situation would have a positive slope and a negative y-intercept? In this activity, students will see a situation involving running a booth at a fair. The person running the booth charges customers a set price per visit (positive rate), but the person running the booth also needs to pay for supplies “up front”—creating a negative y-intercept.

Do Now (time: 5 minutes)
The teacher and students will brainstorm a list of all the types of booths that may appear at a fair.

Note: The booths could feature games of skill, such as a bean bag, ring, or coin toss; knocking down stacked bottles with a tennis ball, putting mini-golf balls, or throwing a ball at a target on a dunking machine. Other booths could feature food concessions, such as selling lemonade, cotton candy, popcorn, snowcones, fried dough, etc. Activity booths could include face painting, karaoke, or photo shoots.

Hook (time: 5 minutes)
The teacher will ask the following questions and have students respond orally, in writing, or by sharing with a partner:

- Do all booths at a fair make money?
- Why would one booth make money and another booth lose money?
What does it mean for a store (or fair booth) to “break even”?
The teacher could prompt the students, noting the cost of the booth construction or rental, set-up costs, supplies and materials, workers at the booth, the need for careful pricing, etc. Then the teacher will ask the students to share their answers with the whole class.

**Presentation** (time: 5 minutes)
Rates can be positive or negative. A \(y\)-intercept can be positive or negative. If we really understand the real-world meaning of slopes and intercepts, we need to recognize when they are positive or negative.

The teacher should provide students with examples of linear situations, such as:

- Calculating wages based on an hourly pay rate
- Calculating the distance a car travels based on the speed of the car
- Calculating the bill for a taxi based on a set fee per mile
- Or, refer to examples from previous lessons—some with positive or negative starting amounts—some with positive or negative (increasing/decreasing) rates

The teacher should model how to determine an equation for both a positive and negative starting amount.

Students should write equations for the remaining linear situations. Then some students can share their equations and other class members can match the equations to the situations. The teacher should consider having students complete a matching game with the situations and equations.

**Practice and Application** (time: 35 minutes)
The teacher will pass out the “At the Fair—A Line is Worth a Thousand Words” Activity Sheet on pp. 4.6.18-4.6.21 in the Supplement, which will allow students to determine profit and loss for two different concession booths: a helium balloon booth and a popcorn booth. The students will explain, calculate, and graph various situations that could occur at each booth. The teacher can read the first situation to the class and the first series of questions.

Students will work with partners and a calculator to complete the “At the Fair—A Line is Worth a Thousand Words” activity. The teacher will circulate among the groups to answer student questions.

**Review and Assessment** (time: 5 minutes)
At the end of the class, the teacher will check for the students’ level of understanding. The teacher will give each student an equation and ask the student to describe a possible situation based on an equation. The teacher and the class will provide feedback. For example:

Given: \(y = 70 + 1.5x\)

**Possible situation:** The temperature started off at 70 degrees and it rose at a rate of 1.5 degrees each hour.

Given: \(y = -5x\)

**Possible situation:** The hiker started at ground level. Each minute hiked, her elevation dropped 5 feet.

**Extension**
Students can complete the extension section of the “At the Fair—A Line is Worth a Thousand Words” Activity Sheet on p. 4.6.21 of the Supplement.
Lesson 5

At the Big E (4 days)

Lesson Goal
Students will put together the skills they have practiced so far to create their own linear models that they can use and make predictions from. Each student will create a booth to be showcased at a fair like the Eastern States Exposition (the Big E). The basis for their model will be the booths they want to run. Students will plan their upfront costs for supplies and rentals (this will comprise their negative $y$-intercept.) Students will decide upon a cost per customer. This will comprise their positive slope.

Lesson 5—DAY 1:

Goal
Students will understand how a profit equation can be created based on upfront costs (which will be negative) and the rate of sales (which will be positive).

Do Now (time: 5 minutes)
The teacher will ask students:

If you could (or had to) run a booth at a fair, list three types of booths that would fit in each category below and explain your reasoning.

The teacher can provide a graphic organizer or have students create the table below.

<table>
<thead>
<tr>
<th>Booths that you think would be MOST profitable:</th>
<th>Booths that you think would make an AVERAGE profit:</th>
<th>Booths that you think would make the LEAST profit:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lemonade Stand</td>
<td>Dunk Tank</td>
<td>Fried Dough</td>
</tr>
</tbody>
</table>

Hook (time: 5 minutes)
The teacher and students will list the “overhead” costs for the following booths. The teacher may provide a graphic organizer or have students create the table below before beginning the activity. Students will likely need examples of overhead costs, so the teacher should model one of the examples first. The teacher and students will then complete one together as a class to ensure that students understand the concept of overhead costs and then students can complete the last two on their own.
A completed box may look like this:

<table>
<thead>
<tr>
<th>Lemonade Stand</th>
<th>Dunk Tank</th>
<th>Fried Dough</th>
<th>Photo Booth</th>
</tr>
</thead>
<tbody>
<tr>
<td>-Cost of lemons</td>
<td>-Rental of the dunk tank</td>
<td>-Cost of ingredients</td>
<td>-Rental of booth/backdrop</td>
</tr>
<tr>
<td>-Cost of cups</td>
<td>-Paying employees</td>
<td>-Paying employees</td>
<td>-Rental of Polaroid camera</td>
</tr>
<tr>
<td>-Cost of water (whether from the tap or bottled)</td>
<td>-350 gallons of water</td>
<td>-Rental of food cart</td>
<td>-Cost of Polaroid film</td>
</tr>
<tr>
<td>-Cost of sugar</td>
<td>-Additional water access</td>
<td>-Utilities</td>
<td>-Paying employees</td>
</tr>
<tr>
<td>-Paying employees</td>
<td>-Cost of tennis balls</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Presentation** (time: 45 minutes)

The teacher will introduce the “My Booth Will Have a Longer Line than Your Booth” Project Packet on pp. 4.6.23-4.6.32 in the Supplement. The teacher and student volunteers will read aloud the first three sections (The Story, Your Assignment, and Rubric) on p. 4.6.24 of the Supplement. Students should highlight or circle important information and take notes or record questions to clarify the expectations of the project. This would be a good time for the teacher to develop an assessment rubric with the students and make adjustments to the criteria based on student needs and classroom expectations. The teacher should also ask students if they have questions or need clarification. Then, the teacher will help the students work through using the information provided to complete the “Booth Comparison Table” section of the Project Packet on p. 4.6.25 of the Supplement.

Students should pair up to give each other feedback on their ideas for the “Your Booth” section, on pp. 4.6.26-4.6.29 of the Supplement. The students will also assist each other to be sure their list of materials and equipment is complete.

Parts 1 and 2:

Why did you choose the ______________ option as your booth? Why does this option seem more interesting or profitable? Can you think of any problems or difficulties you might encounter with this booth?

Part 3:

Have you listed all the upfront costs for your booth? Is there any equipment that you may have missed? Are there any materials or ingredients that may be missing? Have you listed something that isn’t an upfront cost? How many upfront costs are there?

**Lesson 5—DAY 2:**

**Goal**

Today students will continue working through their booth-planning packets:

- Planning your booth
- Researching and finding costs of materials for your own booth
- Creating your equation to represent profit at your booth
- Calculating break-even point
- Graphing your equation and your math teacher’s equation on the same grid
Do Now (time: 5 minutes)

At some point today or tomorrow, the students will have to graph an equation for their booth (and one for their teacher’s booth.)

The teacher will provide students with the graphing pre-assessment found on p. 4.6.22 of the Supplement, which will help the teacher and the student understand the student’s confidence level when graphing equations. The teacher can review graphing skills, if student confidence levels are low.

Directions:

For each graphing skill listed in the table below, circle the most appropriate descriptor of yourself (the student).

At some point today or tomorrow, you’ll have to graph an equation for your booth and one for your teacher’s booth.

<table>
<thead>
<tr>
<th>Graphing Skill</th>
<th>Which best describes you?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deciding on a scale that works for the data</td>
<td>No problem!</td>
</tr>
<tr>
<td></td>
<td>I can do it, but it takes forever!</td>
</tr>
<tr>
<td></td>
<td>It causes me great stress!</td>
</tr>
<tr>
<td>Remembering to label the axes with units</td>
<td>I ALWAYS do that!</td>
</tr>
<tr>
<td></td>
<td>Sometimes I do that.</td>
</tr>
<tr>
<td></td>
<td>What are units?</td>
</tr>
<tr>
<td>Remembering to use a ruler for the axes and lines drawn</td>
<td>Of course!</td>
</tr>
<tr>
<td></td>
<td>Sometimes.</td>
</tr>
<tr>
<td></td>
<td>What ruler? Not really.</td>
</tr>
</tbody>
</table>

Hook (time: 5 minutes)

The teacher will return the “My Booth Will Have a Longer Line than Your Booth” Project Packet to the students from yesterday’s class. The teacher should ask students to review the kinds of booths they chose from the “Your Booth” Section 1, on p. 4.6.26.

The teacher will ask students to sketch the set-up of their booths and label important materials and equipment.

The teacher may wish to provide students with some images of fair booths, in order to get them started on the sketch. Using Google Images, enter “fair booths at the Eastern States Exposition.” There will be ample images, from simple to sophisticated, to choose from. Teachers should also consider allowing students to use computers, colored paper, or graph paper to support visualization of the layout.

Presentation (time: 45 minutes)

Today the students will continue working through the Project Packet (Supplement pp. 4.6.23-4.6.32)

The students will need access to online websites that can provide them with the costs of items they need to purchase or rent for their booth. The teacher may select vetted websites that students may use. For example, local Taylor Rental Party Plus websites allow students to search for the prices of rental items, e.g., a dunking booth for $195.00, tabletop popcorn machine for $55.00. Use websites to check local outlets of stores like Walmart, Stop and Shop, and Big Y to determine the price of food ingredients; find the price of sports equipment online for local firms or nearby chain stores that carry sporting goods, such as Dick’s or Target.
The teacher will review sections 4 (chart and research for items purchased), 5 (chart and research for items rented), and 6 (what you need to charge each person in order to make a profit) with the students. These sections are located on pp. 4.6.26 to 4.6.27. The teachers will also provide directions for researching the cost of the items they need to purchase or rent for their booth.

If students finish in a timely manner, the teacher can ask them to move on to “Make Your Equation” and “Understand Your Equation” on the bottom of p. 4.6.27 in the Supplement.

Lesson 5—DAY 3:

Goal
Students will continue to work through their Project Packet. By the end of this class period, they should have created and graphed equations for their booth.

Do Now (time: 5 minutes)
The teacher will provide each pair of students Post-it notes and a piece of chart paper. The teacher will provide the following instructions:

- Take a Post-it and write a factor that you think will influence how many people visit a specific booth at a fair. Place the Post-it on your chart paper. Take another Post-it—and keeping adding Post-its to your chart paper for as many factors that you can think of.

Students should hopefully come up with:
- Weather/temperature that day
- Weather/precipitation that day
- Accessibility to the booth
  - Is it on the edge of the fair? Is it near some loud speakers?
- Proximity to specific booths
  - Is it in an area of the fair where all of the very popular booths are? Is it near the more “boring” booths?

The teacher can add Post-its to the board to supplement the students’ ideas.

Hook (time: 5 minutes)
The teacher will ask students:

- Which factors are most important? Are they easy or difficult to control?

The students will share their results with the whole class.

Presentation and Work Time (time: 35 minutes)
The teacher will ask the students to continue working with the questions in the “My Booth Will Have a Longer Line than Your Booth” Project Packet, beginning with problem 7 (Make Your Equation) through problem 17, on pp. 4.6.27-4.6.31.

On p. 4.6.31 (in the Project Packet), students will use a spinner to determine the number of visitors at their booth, based the weather each day and the popularity of the booth. They will also need to use the spinner to determine the number of visitors at their math teacher’s booth. Using their equations, the students should calculate both sets of results (problem 17).
Review and Assessment (time: 10 minutes)
Pairs of students should share their equations and the results from problem 17 with the class. Classmates should provide feedback on the equations and the factors that were most important in determining the results.

Students may ask some or all of the following questions:
- What were your independent and dependent variables?
- What were the most important factors that determined the results for your booth?
- What were the most important factors that determined the results for your math teacher’s booth?

Lesson 5—DAY 4:
Goal
Today students will present their results. Will their booths have longer lines than their math teacher’s line? How many visitors will it take for them to break even? Students will find a visual means to present this information that can then be shared with the class.

Rather than informally sharing in class, the students will be making a more structured presentation to their peers. They will select a preferred method of presenting their findings and incorporate some details that will make their presentations interesting to the audience. This low-stakes presentation will help students develop some planning and speaking skills necessary for a later presentation, the “To Buy or Not to Buy a Car” Performance Task.

Do Now (time: 5 minutes)
The teacher will ask student pairs to answer the question in each box below:

<table>
<thead>
<tr>
<th>Which is your preferred way to present material?</th>
<th>Which is your preferred way to receive material?</th>
<th>Which details make information more interesting to you?</th>
</tr>
</thead>
<tbody>
<tr>
<td>-PowerPoint</td>
<td>-PowerPoint</td>
<td>-Humor</td>
</tr>
<tr>
<td>-Story Board</td>
<td>-Story Board</td>
<td>-Visuals</td>
</tr>
<tr>
<td>-Comic</td>
<td>-Comic</td>
<td>-Graphs</td>
</tr>
<tr>
<td>-Other?</td>
<td>-Other?</td>
<td>-Color</td>
</tr>
</tbody>
</table>

Hook (time: 5 minutes)
The student pairs should share their choices with the class. Then, the teacher will pass back the “My Booth Will Have a Longer Line than Your Booth” Project Packet to the students. The teacher and students should read aloud “Conclusion” section of the Packet (p. 4.6.32) that asks students to present their results. The teacher will review the Assessment Criteria with the students and indicate that the presentation is still informal. Each student pair will remember to share the speaking time equally and remember to include all seven bulleted items from the “Conclusion” section in the presentation.
Lesson 6

Important Rates for Car Ownership

At this point, students have increased their knowledge of linear modeling in four real-world situations: analyzing types of pizza slices and window panes, tying knots in yarn, and creating a booth for a fair. Each of these situations added another level of complexity to the previous one, and with each application students were able to take on more independent work. In Lessons 6-8, students will increase their independent learning by researching and interpreting data for another specific linear situation (determining which car is best). They will then use their mathematical skills to create a series of recommendations based on their calculations and present those recommendations to the authentic “interested party” and the class.

Goal

Students will understand that the miles per gallon (mpg) of a car and the cost per gallon of gas are rates (constant rates) that are common knowledge for car drivers. Students will practice solving problems that involve these two rates.

Do Now (time: 5 minutes)

The teacher will ask students to give their best answers to the following questions:

- What is the average price per gallon of gas?
- How many gallons of gas does a typical car hold?
- What is the highest price that a gallon of gas in New England has cost?
- What was the price per gallon of gas when your math teacher started driving?
- How many miles per gallon does the average car get?
- What is a hybrid vehicle?
Hook (time: 10 minutes)
The students will share their answers to the questions above and the teacher can write the answers on the board. Then the teacher will provide the correct answers.

Note: Having an empty gallon jug of milk helps students visualize the size of a gallon of gas. It is also important for the teacher to give students a way (word wall, vocabulary sheet, or note-taking sheet) to organize the information provided during the remainder of the class.

Presentation and Work Time (time: 30 minutes)
The teacher should ensure students understand what “miles per gallon” (mpg) means and what it means to say that we pay for gas “by the gallon.”

The teacher will provide students with some costs per gallon of gas (perhaps from different parts of the United States, different countries or different time periods, so there is variety).

Then the teacher will provide students with the fuel efficiency of some common vehicles (a Prius, a minivan, a school bus, and an Accord would allow for good variety).

Next, the teacher will give students the opportunity to practice calculating fuel costs for various trips:

1. Grandpa drives his minivan 500 miles per month. How much would he pay for the gas needed to do this? What if he could borrow Aunt's Prius—how much would the gas cost then?
2. The family calculated that they will make 3 trips to New York City during the summer. New York City is 240 miles away if you live in Eastern Massachusetts or 150 miles away if you live in Western Massachusetts. How much money would they save if they decided to squeeze into their Honda Accord instead of spreading out in the minivan?

The teacher can also ask the students to think of a place in the United States that they would like to visit. How much would it cost to visit this destination if we took a school bus? Would we save money if we took two minivans?

Note: To help students understand the basic idea of fuel for a car, it can be helpful to use the analogy that fuel is like food for the car. Just as we burn food for energy, cars burn gasoline. We can function for a certain amount of time on a certain amount of food—and then we are hungry and lacking energy. Each car has a specific amount of miles it can go on a certain amount of gasoline (a gallon is the unit we use in the U.S.) So, we have to keep putting gasoline in our cars just as we have to keep eating! Having a gallon size jug will be handy. If it is permissible, having food for the students will be a helpful (and welcome) prop as well. But the teacher needs to check first because it may not be allowed in all programs.

Review and Assessment (time: 10 minutes)
The teacher will ask the students to write their own miles-per-gallon and cost-per-gallon problem and share the problems with the rest of the class.

Note: The students should not “solve” their own problems. The teacher will collect the the problems, and they will be used by the students in Lesson 7.
Lesson 7

Graphs and Rates

**Goal**
Students will practice making graphs to express rates and reading rates from graphs.

**Do Now** (time: 5 minutes)
The teacher will return the miles-per-gallon and cost-per-gallon problems from yesterday. The students will exchange their “make your own problem” with a classmate and solve it.

**Hook** (time: 10 minutes)
The teacher will show students pairs of graphs that have lines that seem to have different steepness to them.

*Note:* The trick is that each set of graphs actually has the same slope—it’s just that very different scales are used for each line. For example, one pair of lines can be showing an hourly wage of $10 per hour. For each graph, hours is on the x-axis and money is on the y-axis. But, if the second graph has a scale for the y-axis that goes up by 10s, its line will be much more gradual than the first graph that might have a scale that goes up by 1s.

Instructions for the students:
- For each pair of graphs, determine the rate that is being illustrated.
- Be sure to include the units for each rate.

Rates can include: mpg, mph, dollars per gallon, dollars per hour, etc.

**Presentation** (time: 25 minutes)
Reading rates from graphs:
The teacher will provide students with more practice reading rates from graphs. They will need to remember that slope is still the rise over run. It is crucial to pay attention to the scale of the axes. When we measure the rise and the run, we need to look at the scale on each axis.

The teacher can continue using an anchor chart that lists the mathematical concepts and their definitions, similar to the sample chart provided in Lesson 1, or the teacher can provide the students with a note-taking sheet or reference sheet for terms and formulas.

*For example:* The rise between two points on a grid is 2 boxes; if each box represents 10 dollars, then the actual rise is $20, based on the rate.

Creating graphs to express rates:
Provide students with some blank grids and sets of rates.

<table>
<thead>
<tr>
<th>Set 1:</th>
<th>mpg</th>
<th>20 mpg</th>
<th>75 mpg</th>
<th>[for 15 gallons]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set 2:</td>
<td>$/hour</td>
<td>$10/hour</td>
<td>$15/hour</td>
<td>[for 20 hours]</td>
</tr>
<tr>
<td>Set 3:</td>
<td>$/gallon</td>
<td>$1/gallon</td>
<td>$1.75/gallon</td>
<td>[for 12 gallons]</td>
</tr>
</tbody>
</table>

Ask students to graph each set of rates. Students will be challenged to find a scale that works for all three members of each set.
Review and Assessment (time: 15 minutes)

The teacher will ask students to play “Telephone Graph Making.” Like a game of “Telephone,” a message (a rate) is being passed from student to student. Hopefully, unlike “Telephone,” the rate stays the same.

Four students will be involved in “Telephone Graph-Making,” and the process has four steps.
(If a group only has three students participating, the student who begins the process with step one could also conclude the process by completing step four. Or, the teacher may participate to add a fourth group member.)

Step One: Each student is given a paper that has alternating grids and “Rates _____” on it. Each student is asked to write their name on the top of the paper and also write a rate on the first “Rate _____” (Be sure they include units). See the “Telephone Graph Making” Activity Sheet on p. 4.6.33 in the Supplement.

Step Two: Then, each student will pass their paper to the next person. That person has to use the next grid to graph the rate that was written above. (Again, be sure they include units.) The students are going to pass their papers again but FIRST, they need to fold over the first “Rate _____” that was written so all that is showing is the graph they just made. Now, when they pass their paper, the next classmate will have only a graph to look at.

Step Three: The next student now interprets the graph and completes the “Rate _____” that is below it. (Yes—remember, units again.) Before passing the paper again, students should fold the paper even further so that only the “Rate _____” that they just completed is showing.

Step Four: The last person has to use the next grid to graph the rate that was written above. (Once more, include units.)

Step Five: Students will unfold the piece of paper to see if the rates did remain intact! If the students find errors, the teacher and other students should indicate where the problem occurred and how the error in graphing or interpreting can be solved. Then the students can return the papers to the student who started them.

If time permits, the “Telephone Graph Making” activity can happen more than once. A repeat of the activity will allow students to practice both graphing and interpreting and improve both skills.

Lesson 8

Getting to Know the Kelley Blue Book

This lesson provides preparation for the Performance Task (Lesson 9).

Goal

Students will recognize that there are both one-time and yearly costs associated with car ownership. Students will also “get to know” the Kelley Blue Book and how helpful it can be for car buying decisions. The teacher may need to project items from Kelley Blue Book for the students to view.
Before starting the learning task, students will need to “interview” a trusted adult who is also a car driver. Students will be using the information from that driver as the basis for their project. The “interview” questions can be found in the “Car Talk” worksheet on p. 4.6.38 of the Supplement.

The teacher will pass out the “interview” form to the students. Each student will need to put their name on the form and the name of a trusted adult (teacher, etc.) that is available to provide the information. The teacher will distribute the interview forms to the trusted adults and ask them to return the information in a day or two. There are only 5 questions on the “interview” form.

**Do Now** (time: 5 minutes)

The students should read the following:

Cars are expensive! Let’s say your dream car for now is a used Honda Civic which has a price tag of $5,500. You have been saving your money since you were 11 years old and you FINALLY have $5,500. You ask a friend to bring you to the used car lot. Your friend looks at you like you’re crazy and says, “Don’t you realize how many costs are involved in owning a car—besides just the price of the car?”

Directions for students:

List as many costs as you can that a potential car owner (driver) must be ready to pay.

Students should come up with gasoline, insurance, sales tax, excise tax, maintenance, and so on.

**Hook** (time: 10 minutes)

The teacher will show the students the quick Pre-Assessment PowerPoint “Car Intro.”

SEE: Pre-Assessment “Car Intro” PowerPoint

The students will write their best guesses to answer the questions on the PowerPoint.

Then the teacher will go over the answers to the questions with the class.

**Practice and Application** (time: 35 minutes)

The teacher will explain the role of the *Kelley Blue Book* (www.KBB.com) for car buyers and sellers. The teacher will ask students to complete the Activity Sheet, “Getting to Know the *Kelley Blue Book.*” See pp. 4.6.34-4.6.37 in the Supplement. If students do not have access to www.KBB.com, this may need to be modified by the teacher by projecting the information or copying the information on a handout.

**Review and Assessment** (time: 5 minutes)

The teacher will ask students to sort the following car costs into one-time costs or yearly costs:

- gasoline
- price of car
- maintenance
- insurance
- excise tax
- sales tax
CULMINATING LESSON
Includes the Performance Task, i.e., Summative Assessment—measuring the achievement of learning objectives

Lesson 9
To Buy or Not to Buy a Car? (4 days)

Students will spend four days researching, analyzing the data, and preparing a visual and written presentation for the following assignment:

Imagine that a trusted adult has asked you to help them decide if they should purchase a newer car or keep the car they already own. It seems that economic conditions are improving and this may be the time to look at newer car options. Using your understanding of linear modeling, you will compare the short- and long-term costs of each vehicle and create a presentation that shows your calculations and provides a detailed recommendation based on the data. It is important that you use the vocabulary of linear modeling for this presentation.

Goal: To determine if it is more economical to buy a newer car or keep the car you already own
Role: A trusted adult has asked you to help them determine whether or not this would be a good time to invest in the purchase of a newer car
Audience: The person(s) who currently owns the car and your classmates
Situation: A trusted adult has owned their current car for several years. The economy seems to have stabilized, gas prices are down, and this may be the time to upgrade their current vehicle.
Product: A presentation/poster/storyboard/PowerPoint that compares the short- and long-term costs of both cars using linear modeling and a recommendation about which option is most economical
Standards: Correct vocabulary and accurate information; story flow

Lesson 9—DAY 1: Beginning the Performance Task

Students will apply their understanding of linear modeling to compare all of the costs involved in buying and driving two cars: one that is already owned vs. one that is being considered. Students will start by brainstorming (and identifying) which car-related costs are one-time (up front) and therefore will comprise the y-intercept and which car related costs are yearly (which will comprise the slope.) Eventually, students will make a recommendation about whether or not buying the newer car is a good economic decision.

Do Now (time: 5 minutes)
The teacher will return the “Car Talk” interviews from Lesson 8 to the students and ask the students to share the information with one other student in the class.

Hook (time: 5 minutes)
The teacher will ask the students:

What is the newer car that your interviewee selected?
How do you think it will compare with your dream car from Lesson 8?
(The answers may reflect fuel economy, cost of the car, maintenance, sales tax, insurance, excise tax, and even the styling or the coolness factor of the car.)

**Presentation** (time: 45 minutes)
The teacher will review the culminating assignment with the students (pp. 4.6.40-4.6.42 of the “To Buy or Not to Buy a Car?” Project Packet). A summary of the project and the outcomes is found on p. 4.6.40, followed by the exemplar final writeup on p. 4.6.41. The criteria for assessment is on p. 4.6.42.

When the class is reviewing the criteria for assessment, the teacher should take the opportunity to create a rubric with the students based on the criteria. The criteria may also be modified to meet the needs of the students and the classroom expectations.

At this point, the teacher should inform the students that this will be a formal presentation, so effective presentation skills will be important as well. The teacher should ask the students to brainstorm some of the characteristics of an effective speaker. The student list should include a variety of vocal and physical delivery skills (projection, articulation, rate/pauses, eye contact, posture, gestures, etc.) The teacher will discuss the delivery skills with the class and indicate that they will have an opportunity to practice these skills before they make their presentations.

The teacher should tell the students that the class will review the assessment criteria and rubric periodically. The students should also feel free to look back at the rubric and assessment criteria to make sure they are meeting all the requirements.

The students should then begin calculating the upfront costs for the vehicles (pp. 4.6.43-4.6.46 in the Supplement). The teacher will circulate among the students as they work, answering questions as needed.

Next the teacher will ask the students to write the introduction to the project and begin their research on p. 4.6.43 of the Project Packet in the Supplement. At the end of the class, the teacher will collect the Project Packets.

**Lesson 9—DAY 2: Continuing the Work**

**Practice and Application** (time: 35 minutes)
The teacher will pass back the Project Packets. The students will complete the upfront costs and move on to yearly costs. The teacher will circulate among the students, answering questions and providing guidance as necessary.

The students should be able to have their equations written for each car by the end of Day 2. Complete the section in the shaded rectangle “Current Car/Potential Car,” at the bottom of p. 4.6.45.

**Review and Assessment** (time: 20 minutes)
If there is time at the end of Day 2, it would be good for students to look at their equations and those of their classmates. They can make estimations about which cars are “worth it” and which aren’t. The teacher should also remind the students to look back at the assignment criteria/rubric to self-assess their work so far. Again, the teacher will collect the Project Packets at the end of class.
Lesson 9—DAY 3: Completing the Work and Practicing the Presentation

Presentation (time: 35 minutes)
The teacher will pass out the Project Packets to the students. During the first half of the class, students will complete data tables, make graphs, and write their conclusions. Then students can prepare their PowerPoint slides/poster/storyboard that includes all of their information. It is important that each student includes a written/verbal/visual recommendation based on their calculations, outlining whether or not the trusted adult should buy the newer car or keep the one they already own (see p. 4.6.47 of the “To Buy or Not to Buy a Car?” Project Packet in the Supplement). The teacher should also review the assessment criteria (p. 4.6.42 of the “To Buy or Not to Buy a Car” Project Packet in the Supplement) and the class rubric to once again ensure that the students have met all the criteria for the project presentation.

Review and Assessment (time: 20 minutes)
The teacher should build in enough time at the end of the class for students to practice their presentations and make their recommendations, which should incorporate visuals and the academic vocabulary of linear modeling.

The teacher should ask the students to refer to the list of characteristics of effective speakers. The teacher can write this list on the board or refer back to the assignment criteria/rubric:

- vocal projection
- articulation
- speaking rate
- eye contact
- posture
- gestures

Students will then practice their presentations with partners and receive feedback. The teacher may want to provide a feedback form so the partners can receive both verbal and visual feedback. Based on the feedback, each student should write an Exit Ticket indicating how they will improve their presentation. The Exit Ticket is located on the top half of p. 4.6.49 of the Supplement.

Lesson 9—DAY 4: Sharing the Assessment

Student Presentations (time: 55 minutes)
At the beginning of the class, the students should take a little time to assemble their materials for the presentations. Then, each student will deliver a presentation and make recommendations to the class. Following each presentation, the teacher and other students will take a few minutes to ask questions or provide constructive feedback.

If time permits, the students can fill out a Feedback Form for each presenter. The Feedback Form will ask students to describe one strength and one area that could be improved. The students can refer to the content information in the presentation or the delivery skills of the presenter. The Feedback Form is on the lower half of “Exit Ticket and Feedback Form” Activity Sheet on p. 4.6.49 of the Supplement.

At the end of the class, the teacher can provide examples from the presentations that illustrate accurate linear modeling and effective presentational skills.

The students will then pass in their Project Packets for grading and feedback.
POST–UNIT REFLECTION

On meeting the Learning and Language objectives
Connections to Empower Your Future
UNIT: Linear Modeling

Future Ready Connections

Teachers are encouraged to use the Future Ready Rubric to evaluate students’ growth and are encouraged to have students self-evaluate their progress using the Future Ready Rubric. Youth have many opportunities to strengthen their communication and listening skills through group discussions, the “Telephone Graph-Making” activity, and their own presentations in Lesson 8. Students will also communicate through writing which can be evaluated for clarity and effectiveness. Youth should also be evaluated for initiative and self-direction as they complete the “Tying the Knot” activities and make their predictions and conclusions, and the Performance Task where they will be responsible for calculating data, graphing, and drawing conclusions.

Teachers should reflect on whether or not youth stay on task without prompting and if they push themselves to thoroughly complete each activity, answer their own questions, and create a detailed final product instead of only addressing the minimum required information. The equation activities are good opportunities to evaluate students on their accountability for completing tasks and if they are actively engaging in critical thinking.

Essential Questions

The Essential Questions from this unit ask:

How can we use mathematical (linear) modeling to make better decisions?

What sorts of situations are linear?

These questions encourage youth to identify and understand how equations and mathematical models have applications outside of the classroom.

Youth can see from Lesson 1 (cutting the party-size pizza), Lesson 4 (booths at the fair) and Lesson 6 (costs related to owning a car) that understanding and utilizing equations and graphs have real-world applications that can affect their budget and their ability to run and participate in a successful business or event.

Teachers can encourage youth to see their mathematical skills as a tool for achieving personal, academic, and professional goals by making further connections to EYF lessons and Future Ready activities and experiences. Teachers may choose to expand on Lesson 6 by incorporating realistic budgets. Teachers can also have youth brainstorm other situations, events, or activities that would benefit from decisions based on linear modeling such as purchasing a house, investments, credit cards, or starting and cultivating a vegetable garden.

Teachers can also make EYF and Future Ready connections to the transfer goals which focus on analyzing data, drawing logical conclusions, and presenting findings in clear and effective ways. The EYF curriculum requires youth to use research and data from self-evaluation tools to make predictions, set goals, and analyze progress. Students will use similar skills found in this unit when they analyze their progress toward their goals and draw conclusions about their next steps and overall progress.

PYD/CRP Connections

The application of math skills to meaningful and purposeful contexts allows for authentic outcomes that youth can practice and duplicate independently in the future.

This unit reflects Culturally Responsive Practice by using realistic situations and problems in the mathematical
equations questions. Students see that math is a universal skill that can be used in the academic, professional, and personal realms. Students can apply their skills to the act of buying a car, slicing a pizza, or planning to run a small business. The connection to the familiar childhood game of telephone during the graph making exercise in Lesson 6 utilizes students’ prior knowledge.

The unit also demonstrates Positive Youth Development by encouraging students to develop and utilize positive relationships with adults in order to complete Lesson 8. Youth also have the opportunity to work with and support each other in the Lesson 6 Review and Assessment activity when they must create their own equations for each other to solve. The responsibility is placed on each youth to demonstrate their own learning and support the learning of their classmates.

**Career Exploration Connections**

*Teachers are encouraged to make additional real-world connections to careers and industries that require mathematical skills.*

Students can brainstorm which career fields and industries rely on employees having strong mathematical skills and then use Massachusetts Career Information System (MassCIS) website to confirm their predictions and assumptions. These fields include:

- Tax prepared
- Chemist
- Pharmacist
- Architect
- Astronomer
- Interior designer
- Carpenter
- Car designer
- Cook
- Urban planner

Youth can also research which post-secondary programs focus on or require math courses and developing math skills.

SEE: MassCIS
https://portal.masscis.intocareers.org

“Students see that math is a universal skill that can be used in the academic, professional, and personal realms.”

*Teachers may also provide biographical information on innovators and famous mathematicians who used their math skills to succeed,* and then ask youth to read and present that information to their peers. Suggestions include Isaac Newton (physicist and mathematician), Fibonacci (introduced Fibonacci sequence), Alan Turing (computer scientist), Albert Einstein (theoretical physicist), Neil DeGrasse Tyson (astrophysicist and cosmologist), Florence Nightingale (statistician and founder of modern nursing), Ellen Ochoa (first Hispanic woman in space), Charles Babbage (considered the father of modern computers), and Ada Lovelace (considered the first computer programmer).

For Technical Assistance with Empower Your Future connections and lessons, please request support by submitting a Coaching Request ticket using the Coaching Feature on TeachPoint.
## Anchor Chart for Mathematical Concepts in Linear Modeling

### Lesson 1

<table>
<thead>
<tr>
<th>Concept:</th>
<th>Definition:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant rate of change</td>
<td>The change in a variable divided by the time it takes (for the change to happen) will always be the same no matter how big a change is measured</td>
</tr>
<tr>
<td>Linearity</td>
<td>Represented by a straight-line graph, the ratio of a change in one variable and the corresponding change in a related variable will always be the same</td>
</tr>
<tr>
<td>Data set</td>
<td>An organized collection of numerical measurements that describe real-world situations or processes</td>
</tr>
<tr>
<td>Graph</td>
<td>A visual means of discovering the relationship between changes in two related variables</td>
</tr>
</tbody>
</table>
Windows Can Be a Real PANE!
Lesson 1

What IS linear? What is NOT linear?

DIRECTIONS: Read the problems and fill out the answers on this Activity Sheet.

1. The A.B. Glare window store has started selling a new kind of window. These windows can be made to order by combining three type of square window panes.

Each pane measures one foot on each side.*
The three types of panes are:

Corner Pane     Edge Pane     Inside Pane

A 3 foot by 3 foot window:

4 corner panes
4 edge panes
1 inside pane

*For the purposes of this problem, we are measuring only the glass, not the frame.

Sketch a 4 foot by 5 foot rectangular window:

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Adapted From: http://www.mathedpage.org/attc/lessons/ch.07/7.02-square-windows.pdf
2. List how many panes of each type were used to make the 4 by 5 foot window in Activity 1.

3. A builder is going to build a cafeteria that will have only square windows. The windows will be made of the panes described in Activity 1. The builder decides to consider various combinations of square windows that will give a total area of exactly 72. For example, two 6 ft by 6 ft. windows would work, because $6 \times 6 = 36$ and $36 + 36 = 72$.

   a. Find a different combination of square windows that will give an area of exactly 72 square feet.
   b. For each window in that combination, find the number of each type of pane the builder will need.

4. To save time when customers ask for square windows, Lara is assembling kits with the correct number of corner panes, edge panes, and inside panes to make square windows of various sizes. Complete this table to show how many panes of each type are needed for a 2 ft. by 2 ft. window; a 4 ft. by 4 ft. window, and so on up to a 10 ft. by 10 ft. window.

<table>
<thead>
<tr>
<th>Square Window</th>
<th>Corner Panes</th>
<th>Edge Panes</th>
<th>Inside Panes</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 ft. by 2 ft.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 ft. by 3 ft.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 ft. by 4 ft.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 ft. by 5 ft.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 ft. by 6 ft.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 ft. by 7 ft.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 ft. by 8 ft.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 ft. by 9 ft.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 ft. by 10 ft.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Adapted From: http://www.mathedpage.org/attc/lessons/ch.07/02-square-windows.pdf
5. Study the table on p. 4.6.3. Describe some of your observations about the pattern of change for each type of pane.

Graphing Square Windows

6. Generalizing:
   a. Explain in words how to find the number of panes of each type in an \( n \) by \( n \) window. Explain each with a reference to a sketch of such a window.
   
   b. Complete the following table by using your generalization in 6a to predict the number of panes of each type that are needed for these larger windows.

<table>
<thead>
<tr>
<th>Square Window</th>
<th>Corner Pane</th>
<th>Edge Pane</th>
<th>Inside Pane</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 by 20</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30 by 30</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50 by 50</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 by 100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n ) by ( n )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. Add up the algebraic expressions for the number of each type of pane. If you did your work correctly, the sum should be a simple one—as it related to the square window dimensions. What does your result represent in the context of the windows?

Adapted From: http://www.mathedpage.org/attc/lessons/ch.07.02-square-windows.pdf
8. On the same set of axes use this grid to:

   a. Graph the number of corner panes as a function of the length of the side of the window.
      For example, a 3 ft. by 3 ft. window used 4 corner panes. The point (3, 4) would be on your graph.

   b. Graph the number of edge panes as a function of the side length.

   c. Graph the number of inside panes as a function of the side length.

9. Study your graph. Which one(s) are linear? Which one grows the fastest?

10. CHALLENGE:
    Assume that you try to make any number of square windows, with the goal of having as few panes as possible left over.
    
    a. If you start with 100 panes of each type, what size windows should you make? What will be left over?
    
    b. Compare your answers with other students’.
Measurement Using Conventional or Metric Rulers
Lesson 2—Pre-Assessment

A conventional U.S. ruler is twelve inches long, or one foot. The units of the ruler are:
1/16” (the smallest unit)
1/8”
1/4”
1/2”
1 Inch
1 Foot (largest unit)

Think—Using a conventional U.S. ruler, begin at zero and measure the following parts of a pencil. Enter the measurements in the data table provided using whole numbers and fractions:

1. Length of the entire pencil (end to end)
2. Length of the metal ring that holds the eraser on to the wooden part of the pencil
3. Diameter of the eraser
4. Greatest length of the sharpened wooden cone, not including the graphite tip
5. Width of the graphite lead where it first exits the wood
6. Width of one of the six wooden planes on the pencil

<table>
<thead>
<tr>
<th>Variable</th>
<th>My U.S. data</th>
<th>Partner’s U.S. data</th>
<th>My metric data</th>
<th>Partner’s metric data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entire pencil</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Metal ring</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eraser diameter</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sharpened cone</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Graphite lead</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wooden plane</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A metric ruler is based on a system of ten. The units on the ruler are millimeters (the smallest unit), centimeters, and a meter (the largest unit).

10 millimeters = 1 centimeter
100 centimeters = 1 meter

Now complete the exercise once again. Using a metric ruler, begin at zero and measure the following parts of a pencil. Enter the measurement in the data table on p. 4.6.6 using decimals to the nearest millimeter:

1. Length of the entire pencil (end to end)
2. Length of the metal ring that holds the eraser on to the wooden part of the pencil
3. Diameter of the eraser
4. Greatest length of the sharpened wooden cone, not including the graphite tip
5. Width of the graphite lead where it first exits the wood
6. Width of one of the six wooden planes on the pencil

Pair—Exchange pencils with a partner. You will complete each exercise using the U.S. ruler and the metric ruler and enter your findings on your partner’s data table.

Compare your answers. If there are differences in your recorded data, measure for that data together. If you need to, use a magnifying glass to get the best results. What were your findings?

Share—two or three of the differences you found with the class.

How did re-measuring help you with accuracy?
Which type of ruler scale (U.S. or metric) is easier to use? Which one do you feel is more precise?

Extension—Explain the meaning of the carpenter’s expression, “Measure twice—cut once. Measure once—cut twice.”

What are some examples of other occupations where measuring accurately is important? The answers can include other types of measurements besides linear measurements.
Tying the Knot (When Did You Learn to Tie Your Shoes?)
Lesson 2

1. Measure the length of your yarn to the nearest tenth of a centimeter.
Then begin tying knots in your yarn and recording the lengths as you go. Record all the data in the table.

<table>
<thead>
<tr>
<th>Number of knots</th>
<th>Length of yarn in cm.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

2. On the grid above, make a scatterplot comparing the number of knots (x) to the yarn length (y).

3. Draw in the best fit line for this scatterplot.

4. Find an equation that relates the number of knots (x) to the length of the yarn (y). ________________
   Show or explain how you found your equation and circle your equation.

5. What “family of functions” does this graph and equation belong to? ________________________________
   How do you know?

6. Where does your graph hit the y-axis? _________ What does this mean in the context of the problem?

7. Where does your graph hit the x-axis?__________ Why does this point have no real-world meaning?

8. What is the rate of change of the yarn length as you increase the number of knots? ________________
9. Suppose you tied 10 knots in your yarn. How long would you predict the string would be? __________
   How did you get this answer?

10. If you started with a longer piece of yarn, how would your equation change? Explain why.

11. If I had a thicker piece of yarn, the same length as yours, to work with, how do you think my equation
    would compare to yours? Be specific and explain why/how it’s different.

12. Lacy Zuntide decides to do this same activity with a different piece of yarn. Unfortunately, she lost her
    paper. She remembers that when there were 4 knots the yarn was 233 cm long and she started with
    265 cm of yarn.

   a. Using this information, write an equation that would predict the length of yarn for any given number
      of knots. Show or explain how you came up with your equation.

   b. Use this equation to predict the length if Lacy tied 15 knots in her yarn. Show your work.

   c. Lacy measured her yarn at one point and it was 65 cm long. How many knots did she have then?
      Show your work.
13. May Kinot also repeated this yarn activity and lost her paper. She remembers that when she had 4 knots the piece of yarn was 305 cm long and when she had 9 knots the yarn was 255 cm long. She does not remember how long the yarn was when she started.

   a. Using this information, write an equation that would predict the length of yarn for any given number of knots. Show your work.
   
   b. What was the original length of May’s piece of yarn? Show or explain how you found this.
   
   c. If May tied 20 knots, how long should the yarn be? Explain how you found this answer.

14. How can you tell from the equations who had the longer yarn to start with, May or Lacy? (Who started with a longer yarn and how do you know?)

15. How can you tell from the equations who had the thicker yarn, May or Lacy? (Who was using a thicker yarn and how do you know?)

16. Moe R. Tuteye’s equation for his piece of yarn is \( y = -9x + 432 \).

   a. What is the original length of his piece of yarn? ____________

   b. How many knots does he need to tie to get a length of 306 cm? Show your work.

   c. How many knots does he need to tie to get a length of 225 cm? Show your work.

   d. What is the greatest number of knots he can tie in his piece of yarn? Can you be sure of this answer? ____________ Explain.
$y = mx + b$

Lesson 2—Extension

DIRECTIONS:
For this lab, you will be writing linear equations in this form, $y = mx + b$.
Your goal is to find out how the $m$ and the $b$ affect what type of line is drawn. You will need to use the:

SEE: Desmos Graphing Calculator | www.desmos.com/calculator

1. Type in each of these equations to see what the computer draws.
   Use a different “$y =$” for each equation so all three are graphed on the same grid.
   a) $y = 3x - 2$
   b) $y = 0.5x - 2$
   c) $y = -6x - 2$

2. Sketch the three lines that were drawn:

3. What point do these graphs have in common?

4. What part of the equation tells you that they will have this point in common?

5. What is the main difference between these graphs? (Use complete sentences in your answer.)

6. What part of the equation tells you that they will have this difference?
7. Now, clear your grid and enter these four equations. Again, use a separate “y = …” for each equation.
   a) \( y = 2x + 4 \)  
   b) \( y = 2x + 1 \)  
   c) \( y = 2x \)  
   d) \( y = 2x - 5 \)

8. Sketch the four lines that were drawn:

9. Do these lines cross the y-axis at the same point?  

10. How could you tell the answer to Question 9 JUST by their equations?  

11. What is the same about each of the graphs above?  

12. What part of their equations tell you that they will have this similarity?  

13. A linear equation is a rule (a function) that can be written in this form: \( y = mx + b \)
   The values of \( m \) and \( b \) affect the tilt and position of the line that is drawn.
   So far, what do you know? Answer the following:
   
a. Which of these (\( m \) or \( b \)) affects how STEEP or FLAT the line is?  
b. Which of these (\( m \) or \( b \)) affects the position of the line? (That is, where the line crosses the y-axis?)  
c. Which of these (\( m \) or \( b \)) do you think is called the “slope” of the line?  
d. Which of these (\( m \) or \( b \)) do you think is called the “y-intercept” of the line?
14. Think of three equations (of your own creation) that will have **DIFFERENT SLOPES** but the **SAME Y-INTERCEPT**. List your equations below and sketch all three on the graph at right.

   Equation 1 ________________________________

   Equation 2 ________________________________

   Equation 3 ________________________________

15. Think of three equations (of your own creation) that will have **SAME SLOPES** but **DIFFERENT Y-INTERCEPTS**. List your equations below and sketch all three on the graph at right.

   Equation 1 ________________________________

   Equation 2 ________________________________

   Equation 3 ________________________________

16. Have some fun with the $m$ value:

   a. With the help of the Desmos Graphing Calculator, find an equation for a line that will be very steep. Once you know your equation is correct, list your equation and sketch it on the graph at right:

      Equation ________________________________

   b. With the help of the Desmos Graphing Calculator, find an equation for a line that will be almost horizontal (flat). Once you know your equation is correct, list your equation and sketch it on the graph at right:

      Equation ________________________________

   c. With the help of the Desmos Graphing Calculator, find an equation for a line that will tilt down (like this $\rightarrow \downarrow$). Once you know your equation is correct, list your equation and sketch it on the graph at right:

      Equation ________________________________
17. Consider this linear equation: $y = 2x + 5$

a. Write an equation for a line that will be steeper: ____________________________

b. Write an equation for a line that will be even steeper than problem (a), but slants the opposite way:

   ____________________________

c. Write an equation for a line with the same steepness as problem (a) but crosses the $y$-axis below the origin:

   ____________________________

d. Write an equation for a line with the same steepness as problem (a) but crosses the $y$-axis at an even lower point than your answer to problem (c):

   ____________________________

18. Explain what $m$ and $b$ in this equation control about a line: $y = mx + b$. Use complete sentences.

   a. What does the $m$ number control?

      _____________________________________________

      1. If the $m$ value were big—like 5 or 10 or 15, what would happen to the line?

      2. If the $m$ value were a fraction—like $\frac{1}{2}$, what would happen to the line?

      3. If the $m$ value were negative—like -3 or -$\frac{1}{2}$, what would happen to the line?

   b. What does the $b$ number control?

      _____________________________________________

      1. What would have to be true about the $b$ value for the line to cross the $y$-axis below the origin?

      2. What would have to be true about the $b$ value for the line to cross the $y$-axis right at the origin?
19. Have your teacher check your answers to #17, and get the teacher’s initials: ______________________

20. Use it! Turn your computer off and try the task below.

Below are a series of equations. Your task is to sketch the line for each equation on the grid that accompanies the equation.

\[
\begin{align*}
y &= 5x - 3 \\
y &= -5x - 3 \\
y &= 2x \\
y &= 0.2x + 1 \\
y &= -\frac{1}{2}x + 5 \\
y &= x \\
y &= -3x - 1 \\
y &= 0.01x
\end{align*}
\]
Still Tying the Knot (Ever Been to a Wedding?)
Lesson 3

DIRECTIONS: Read the problems and fill out the answers on this Activity Sheet.

1. We’ve learned about linear equations in this form: \( y = mx + b \) (or \( y = ax + b \))
   a. What does the value of \( m \) (or the \( a \)) tell you about a line?
   b. What does \( b \) tell you about a line?
   c. How can you prove that a point does or doesn’t fall on a specific line?

2. In the “Tying the Knot” activity, we came up with an equation like this: \( y = -3x + 27 \)
   a. This is a linear equation, right?
   b. What does \( x \) represent in this situation?
   c. What does \( y \) represent in this situation?
   d. What is the \( m \) (or \( a \)) value in the equation above?
      In terms of the “Tying the Knot” activity we were doing, why is the \( m \) value negative?
   e. What is the \( b \) value in the equation above?
      In terms of the “Tying the Knot” activity we were doing, why is the \( b \) value positive?
   f. The point (4, 15) would be a point on the line of the equation above.
      What is the real world significance of this point—in terms of this activity?
   g. Would the point (7, 2) be a point on this line? Prove your answer mathematically:
3. We say that linear equations make straight lines. Lines are straight because they have a constant rate of change. That is, the steepness between any two points on that line is the same.
   a. Which part of a linear equation represents the constant rate of change? ______________
   b. What did we do during the “Tying the Knot” activity in order to try to make the rate of change stay constant?

4. During a recent meeting, Evergreen School teachers tried the “Tying the Knot” activity. Here are the equations that the teachers wrote:

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ms. Dolci</td>
<td>( y = 80 - 9x )</td>
</tr>
<tr>
<td>Mr. McSweeney</td>
<td>( y = -5x + 90 )</td>
</tr>
<tr>
<td>Mr. Newman</td>
<td>( y = -7x + 80 )</td>
</tr>
<tr>
<td>Ms. Zelaya</td>
<td>( y = -1x + 30 )</td>
</tr>
<tr>
<td>Ms. Gomes</td>
<td>( y = 45 - 5x )</td>
</tr>
</tbody>
</table>

a. Who was using the thinnest yarn? __________ How do you know?

b. Who was using the longest yarn? __________ How do you know?

c. Which two teachers were using the same type of yarn? __________ How do you know?

d. Which two teachers started with the same length of yarn? __________ How do you know?

e. Who will be able to tie the most knots in their yarn? __________ How do you know?

f. The teachers graphed their equations! Unfortunately, no one made a key or labeled their lines.

**Your task:**
Match each line with the correct teacher’s name.

Purple line __________
Green line __________
Red line __________
Orange line __________
Black line __________
At the Fair—A Line is Worth a Thousand Words
Lesson 4

DIRECTIONS: Read and perform the problems on this Activity Sheet. Please show your work and/or explain any numerical answer.

You have the opportunity to run a helium balloon concession at the Adams Vendor and Craft Fair next spring. In order to sell balloons, you must pay a $20 fee to the concession sponsors and buy $50 worth of balloons and helium. You can sell each balloon for $.80. You need to calculate your net profit for the fair. Remember, profit is the money you bring in minus the expenses you have to pay out.

1. How much profit will you have if you sell
   a. 150 balloons $____________________
   b. 200 balloons $____________________
   c. 500 balloons $____________________
   d. 50 balloons $____________________
   e. 80 balloons $____________________

2. Do your answers to 1d and 1e seem strange? Explain them.

3. How many balloons must you sell to make a profit of
   a. $40 _________________
   b. $100 _________________

4. How many balloons would be sold to post the following losses:
   a. $20 _________________
   b. $50 _________________
   c. $100 _________________
At the Fair—A Line is Worth a Thousand Words
Lesson 4

5. Use the information you found in problems 1, 3, and 4 to complete this table.

<table>
<thead>
<tr>
<th>Balloons Sold</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<tr>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

6. Use information from the table you completed in problem 5 (above) to make a graph that illustrates the situation. Remember to include labels, units, title (use graph paper provided by the teacher).

7. Do your data points line up on the graph? ________________ Why should they?

8. When you sell 0 balloons, your profit is not 0. Explain why.

9. If you double the number of balloons you sell, do you double your profit? ________________
   Explain why or why not.

10. What is the number of balloons you need to sell to break even (have zero profit or loss)? ____________
    How can you find this answer?

11. What is the real-world meaning of the y-intercept for this specific situation?
12. If you know how much profit you make on 100 balloons, how can you use that information to get the amount of profit for 101 balloons? What about for 99 balloons?

13. Write an equation that relates number of balloons sold \(x\) to profit \(y\).

14. At some point this equation will no longer make sense. For example, can we use this equation to figure out how much profit you will make if you sell 5,000 balloons? Explain why not.

15. Another concession you could run at the fair is a popcorn booth. You paid $40 for the food vendor fee, rented the popcorn machine for $60, purchased $20 worth of supplies, and then sold the popcorn for $1 per bag.

   How much profit will you have if you sell:
   a. 150 bags $____________________
   b. 500 bags $____________________
   c. 50 bags $____________________
   d. 80 bags $____________________

16. Come up with 5 more sets of data for the popcorn and then graph this relationship on the same graph you made for the balloon problem. Again, the data should line up.

   Start by listing the 5 new sets of popcorn data here:

<table>
<thead>
<tr>
<th>Data Set</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
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<td></td>
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<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
17. What is the number of bags of popcorn (Problem 15) that you will need to sell in order to break even?

18. Which should you sell, balloons or popcorn, if you think you will only sell 175 items? Justify your answer:

19. Write an equation that relates number of bags of popcorn sold \((x)\) to profit \((y)\).

20. There are a number of items that you can sell where it won’t matter which you are selling and you would earn the same profit either way.
   a. What number of items is it? __________________________

   b. Did you use the graph or equations to answer this question? __________________________
      If you used the equations, your work should be shown under problem (a) above.
      If you used the graph, explain how you used it.

EXTENSION

Ms. Kusmeskus is going to work at the fair too. She’s going to do face painting. She has calculated that if 200 children get their faces painted, she’ll make a profit of $325. She also determined that if 90 children get their faces painted, she’ll make a profit of $105.

   a. Write an equation in point-slope form for how much profit \((y)\) she gets depending on the number of little faces she paints \((x)\):

   b. Convert your equation to slope-intercept form. What is the real-world meaning of the \(y\)-intercept in this situation?
Graphing Skills Pre-Assessment
Lesson 5

**DIRECTIONS:** For each graphing skill below, circle the most appropriate descriptor for yourself (the student).

<table>
<thead>
<tr>
<th>Graphing Skill</th>
<th>Which best describes you?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deciding on a scale that works for the data</td>
<td>No problem!</td>
</tr>
<tr>
<td></td>
<td>I can do it, but it takes forever!</td>
</tr>
<tr>
<td></td>
<td>It causes me great stress!</td>
</tr>
<tr>
<td>Remembering to label the axes with units</td>
<td>I ALWAYS do that!</td>
</tr>
<tr>
<td></td>
<td>Sometimes I do that.</td>
</tr>
<tr>
<td></td>
<td>What are units?</td>
</tr>
<tr>
<td>Remembering to use a ruler for the axes and lines drawn</td>
<td>Of course!</td>
</tr>
<tr>
<td></td>
<td>Sometimes.</td>
</tr>
<tr>
<td></td>
<td>What ruler? Not really.</td>
</tr>
</tbody>
</table>
My Booth Will Have a Longer Line than Your Booth
Lesson 5

Project Packet

Contents:
The Story and Your Assignment
Booth Comparison Table
Your Booth
  Plan Your Booth
  Determine Your Costs
  Make Your Equation
  Understand Your Equation
  Compare Your Profits
  Day of the Fair—Sell Your Products!
  Spinner Role Key
Conclusion

Name: ________________________________
Date: __________________________________

WIDGET TOSS
5 plays for $2
My Booth Will Have a Longer Line than Your Booth
Lesson 5

The Story: Local Flavor
Every year, the Big E comes to West Springfield, MA. It is an event (a fair) that is three weeks long and is billed as “New England’s Great State Fair.” Each booth charges people to eat, buy a craft, play a game, or have a turn at a fun activity. Some examples include pony rides ($10 each), pizza by the slice ($3 per slice), climbing wall ($10 per 5 minutes of climbing), homemade cookies ($1.50 each), pint of strawberries ($4 per pint), and face-painting ($2 each).

Your Assignment
a. Research a booth idea of your own.
b. Create a linear equation that represents the profit you would earn for running your booth.
c. Compare your booth to your math teacher’s booth using a system of equations and a graph.
d. Answer the questions and draw conclusions regarding how your profit will compare to your math teacher’s profit.

Assignment Points

<table>
<thead>
<tr>
<th>Criteria</th>
<th>On-time</th>
<th>Effective Use of Class Time</th>
<th>Packet Accuracy</th>
<th>Packet Presentation</th>
<th>Graph</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max Points</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td>5</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>Points Awarded</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total Points</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
# My Booth Will Have a Longer Line than Your Booth

## Lesson 5

**DIRECTIONS:** Use the information provided to complete this Booth Comparison Table.

<table>
<thead>
<tr>
<th>Title and description of booth</th>
<th>Ms. S’s booth</th>
<th>YOUR math teacher’s booth:</th>
<th>Ms. T’s booth</th>
<th>Mr. Z’s booth</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Foam Sword Fighting!</strong></td>
<td>Opponents stand on stools and use foam swords to try to knock their opponent off their stool.</td>
<td><strong>Fidget Toss!</strong> You throw the lopsided fidgets at a game board with different point values. You get 5 tries and anyone who gets over 100 points wins a prize.</td>
<td><strong>Climbing Wall!</strong> Put on your harness, attach to the belay line… and climb away!</td>
<td><strong>mmm… π!</strong> The most delicious assortment of home-baked π on this side of the Mississippi River!</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sketch of booth</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Expenses</th>
<th>Renting a space at the Big E: $250</th>
<th>Renting a space at the Big E: $250</th>
<th>Renting a space at the Big E: $250</th>
<th>Renting a space at the Big E: $250</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>• Swords cost $3 each (need 4): $12</strong></td>
<td><strong>• Each fidget costs $0.50 to make</strong></td>
<td><strong>• Renting climbing wall for two days (includes belay ropes): $175</strong></td>
<td><strong>• Ingredients for the π!</strong></td>
<td><strong>• Renting refrigerator</strong></td>
</tr>
<tr>
<td><strong>• Stools to stand on cost $10 each (need 2): $20</strong></td>
<td><strong>• Your math teacher needs 30 fidgets</strong></td>
<td><strong>• Renting harnesses and helmets: $10 each</strong></td>
<td><strong>• Need 4 of each (harnesses and helmets)</strong></td>
<td><strong>• Renting refrigerator</strong></td>
</tr>
<tr>
<td><strong>Total Cost: $32</strong></td>
<td><strong>Total Cost: $39</strong></td>
<td><strong>Total Cost: $255</strong></td>
<td><strong>Total Cost: $__________</strong></td>
<td><strong>Total Cost: $__________</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cost per customer</th>
<th>$5</th>
<th>$3</th>
<th>$__________</th>
<th>$3 (per slice)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit equation</td>
<td>( y = -282 + 5x )</td>
<td>( y = ____________ )</td>
<td>( y = -505 + 10x )</td>
<td>( y = -430 + 3x )</td>
</tr>
</tbody>
</table>
My Booth Will Have a Longer Line than Your Booth
Lesson 5

YOUR BOOTH

Plan for Your Booth:
1. Brainstorm ideas for a booth that YOU would like to run. List at least 3 of your ideas:

   1. 
   2. 
   3. 

2. Circle the booth idea you like best.

Determine Your Costs:
3. List out all the upfront costs you will have for this booth. (For example: rental costs, materials needed, and items rented—such as a popcorn machine or refrigerator, and so on.)

4. Fill out the chart below for items to be PURCHASED (not rented) and go online to find each price.

<table>
<thead>
<tr>
<th>Item that needs to be purchased:</th>
<th>Price of each item needed for weekend:</th>
<th>How many of these are needed?</th>
<th>Total cost for this material:</th>
<th>Homepage of the website where price was found:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Subtotal of purchased items
My Booth Will Have a Longer Line than Your Booth

Lesson 5

5. Fill out the chart below with items needed that are RENTED (not purchased).

<table>
<thead>
<tr>
<th>Item that needs to be rented:</th>
<th>Price of each item needed for weekend:</th>
<th>How many of these are needed?</th>
<th>Total cost for this material:</th>
<th>Homepage of the website where price was found:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Subtotal of rented items

Total cost of purchased items

Total cost of rented items

TOTAL UPFRONT COSTS FOR BOOTH

6. What will you charge each person who visits your booth? ____________

Make Your Equation:

7. Use your answers above to make a linear equation for the total profit you will earn \( y \) if \( x \) number of people visit your booth: ________________ (Remember, your upfront costs are NEGATIVE and need to be paid in order to calculate profit.)

Understand Your Equation:

8. Look back at your equation. Should the \( y \)-intercept be positive or negative? ________________ Why?
My Booth Will Have a Longer Line than Your Booth
Lesson 5

9. Look back at your equation. Should the slope be positive or negative? ________________ Why?

10. If NO ONE visits your booth, what will your profit be?

11. If 100 people visit your booth, what will your profit be?

12. In 2015, the weather was BEAUTIFUL the whole weekend of the local fair. In fact, it was estimated that 2,000 people visited the fair! Let’s say that the weather is great again this year and 2,000 people come again. Since not everyone will visit your booth, let’s say that 50% of the people who come to the fair come to your booth.
   a. How many people does that mean will visit your booth? ________________
      (Show your work.)

   b. Use your equation to determine how much profit you would make. ________________
      (Show your work.)
13. Vendors (people who sell stuff) like to know their “break-even point.” This is the number of items that need to be sold in order for them to just cover their costs.
   a. What will the profit be at the “break-even point”? ___________

   b. Use your equation to determine how many people need to visit your booth in order for you to “break even.”

   c. Use these answers to write a good sentence about what you just found out. For example, “If __________ people come to my booth, I will __________.”

Compare Profits:

14. Let’s say that your booth is right next to your math teacher’s booth. (Refer to the first page of this packet for his/her equation.)
   a. Write your 2 equations as a system.

   b. Use substitution to solve this system.

   c. Use complete sentences to explain what your solution means in terms of people and profit:
15. Use graphing to solve the system of equations you wrote. (On a separate piece of graph paper)
   a. Make a FULL PAGE, FOUR QUADRANT graph that goes up to 1,500 people visiting your booth and goes from the y-intercept up to $5,000.
   b. Clearly mark the location of the solution to the system on the graph.
   c. Include a key that has the equation and name of each booth.

The Day of the Fair
We never know how many people are going to come to the fair. If the weather is great—lots of people will be there. But, if the weather is rainy, there won’t be as many people. Also, we can’t really predict how popular your booth will be. Use a spinner to determine these factors for each day of the weekend.

16. Show your work as you use the results of your spins to determine how many people will visit each of your booths (for the entire weekend). Use the Spinner Key on the next page (p.4.6.31) for guidance to tabulate the spinner results. Put your answers in the table below.

<table>
<thead>
<tr>
<th></th>
<th>Saturday</th>
<th>Sunday</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Your booth</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Your math teacher's booth</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Spinner Key:

<table>
<thead>
<tr>
<th>Spin #1</th>
<th>Spin #2</th>
<th>Spin #3</th>
<th>Spin #4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The weather on Saturday</strong></td>
<td><strong>The weather on Sunday</strong></td>
<td><strong>Percent of people at the fair who visit your booth each day</strong></td>
<td><strong>Percent of people at the fair who visit your math teacher’s booth each day</strong></td>
</tr>
<tr>
<td>Spin</td>
<td>What it means:</td>
<td>Spin</td>
<td>What it means:</td>
</tr>
<tr>
<td>1</td>
<td>Hurricane conditions! Only 200 people attend the fair.</td>
<td>1</td>
<td>Your booth is not very popular. Only 10% of the visitors come to your booth.</td>
</tr>
<tr>
<td>2</td>
<td>Light rain all day. Only 500 people brave the weather and come to the fair.</td>
<td>2</td>
<td>Your booth is not very popular. Only 10% of the visitors come to your booth.</td>
</tr>
<tr>
<td>3</td>
<td>Light rain all day. Only 500 people brave the weather and come to the fair.</td>
<td>3</td>
<td>Your booth is somewhat popular. 30% of the visitors come to your booth.</td>
</tr>
<tr>
<td>4</td>
<td>Cloudy with a chance of rain all day. 1,500 people come to the fair</td>
<td>4</td>
<td>Your booth is fairly popular. 40% of the visitors come to your booth.</td>
</tr>
<tr>
<td>5</td>
<td>Gorgeous day!! 2,500 people come to the fair!</td>
<td>5</td>
<td>50% of the visitors come to your booth!</td>
</tr>
<tr>
<td>6</td>
<td>Gorgeous day!! 2,500 people come to the fair!</td>
<td>6</td>
<td>Your booth is the MOST popular one at the fair! 60% of the visitors come to your booth!</td>
</tr>
</tbody>
</table>

17. Now, use your two equations to determine each of your profits for the weekend. Be organized and clear as you calculate these two amounts.
My Booth Will Have a Longer Line than Your Booth
Lesson 5

Conclusion: Use a PowerPoint, story board, or comic to tell the story of your booth at the Big E.

Include:
- The big idea that you’re going to run a booth at the Big E.
- Setting up your booth and the upfront costs connected to that.
- The equation for your booth, with each part explained.
- The equation for your math teacher’s booth, with each part of the equation explained.
- How many visitors it will take for each of your booths to “break even.”
- Explain what this phrase means:
  The weather, fair attendance, and number of visitors at each of your booths for the first weekend of the fairs (this will be the results from your spinner)
- And, who will make more money? Explain how you determined that.
Telephone Graph-Making
Lesson 7

Name: ____________________________

Rate: ____________________________

Rate: ____________________________

Rate: ____________________________
Getting to Know the *Kelley Blue Book*

Lesson 8

Today, you will learn to use a website that helps millions of adults know much more about cars and their values than you might have thought possible. Although you may not be buying a car anytime soon, you will need to be able to use this website as you complete the upcoming project.

Here’s the web address:  www.kbb.com

**The Trade-in (so you have an old car you want to trade in):**

Let’s say you have a 2002 Honda Civic. Its mileage is already 100,000 miles. You’re thinking of trading it in to get some money toward a newer car. Here’s how you can use *Kelley Blue Book* to find out what he trade-in value of your car is:

1. On the left sidebar, click on the second green rectangle “Check My Car’s Value.”

2. Input the following information:

   - **Year:** 2002  
   - **Make:** HONDA  
   - **Model:** CIVIC  
   - **Mileage:** 100,000 miles

3. If the website asks you to input your zip code, put in the zip code for Springfield, MA: 01101

4. Click on the first car option. Coupe: Sleek and Sporty 2 Door

5. Let’s say your car is the DX Coupe 2D model. Click on the orange rectangle that says “Choose this style”

6. Leave all of the standard equipment options that are pre-checked, scroll down, and click on the orange rectangle that says “See Blue Book Value.”

7. Click on the button that says “Trade In to a Dealer.”

8. Let’s say that your car is in good condition. Scroll down to that condition, and click on the orange rectangle that says “Get Blue Book Value.”

   **How much can you trade in your car for?**

9. Now, let’s change some information. At the top of the page, change the mileage from 100,000 to 20,000 and click on the “Change” rectangle. Under the new value that you see, change the “Good Condition” to “Excellent Condition” in the dropdown menu.

   **How much can the car be traded in for now?**
Getting to Know the *Kelley Blue Book*
Lesson 8

What if you had a super fancy 2012 BMW (in the 7 series) to trade in!

1. Follow the same steps as you did for the Civic—but this time plug in the information about a 2012 BMW 7 series. We’ll say the BMW has only 10,000 miles on it and that it is the 740i Sedan 4D. Leave the standard equipment pre-selected options checked and say that you will trade it in to a dealer.

   What is the trade-in value if it’s in “EXCELLENT” condition? _______________________

   What is the trade-in value if it’s in “FAIR” condition? _______________________

2. Experiment with the “options” that are offered on a BMW.

   Find one option that changes the trade-in value by A LOT. Which option? ___________________

   Find one option that barely changes the trade-in value. Which option? ___________________

Buying a NEW CAR: Now, let’s say you want to buy a NEW CAR...

1. Go back to the www.kbb.com homepage.

2. Click on the green rectangle that says “Price New/Used Cars.”

   Click on “I ALREADY KNOW WHAT KIND OF CAR I WANT.”

3. Input the following information:

   Make: Dodge   Model: Charger

4. Select the newest model Dodge Charger by clicking the orange rectangle.

5. Select the SE model by clicking on the orange rectangle that says “Choose this style.”

6. Click the white rectangle that says “Price with standard options.”

7. What is the base MSRP of a new Dodge Charger? _______________________

8. Go back to the options page and click on all of the “fancy” features available.

   What is the price if the car is “fully loaded”? That is, if it has lots of fancy features? _______________________

9. In the left sidebar, find the rectangle that says “Specs.”

10. What is the Charger’s fuel efficiency (mpg)

   on the highway? ___________________ in the city? ___________________ combined? ___________________
11. Besides fuel economy, what are 4 other pieces of information listed under “specifications” or “specs”?

1. _____________________________________________________________________________
2. _____________________________________________________________________________
3. _____________________________________________________________________________
4. _____________________________________________________________________________

Buying a USED CAR: Now, let’s say you want to buy a USED CAR...

1. Go back to the www.kbb.com homepage.

2. Click on the green rectangle that says “Price New/Used Cars”

3. Input the following information:
   - Year: 2000
   - Make: Chevrolet
   - Model: Corvette

4. Select the Coupe: Sleek and Sporty 2-Door option

5. Select the Hard Top 2D option and click the orange rectangle that says “Choose This Style”

6. Put in the Mileage: 100,000 miles and click the orange rectangle that says “Choose price type.”

7. When you buy a used car, you can buy it from a private person (someone who just wants to sell the car they have) OR you can buy it from a car dealer.
   - Select the “Buy From a Dealer” option. Click the orange rectangle that says “Get Used Car Price.”
   - What price will a car dealer ask you to pay? _____________________________________________________________________________

8. Find the “specifications” of the Corvette by clicking on the blue tab that says “Specs.”

9. What is the Corvette’s fuel efficiency (mpg):
   - on the highway? _______________ in the city? _______________ combined? _______________
Questions:

1. How comfortable do you feel using this website to …
   a. Find the trade-in value of a car? (circle)  Very  Somewhat  Not-so-good
   b. Find the cost of a new car? (circle)  Very  Somewhat  Not-so-good
   c. Find the cost of a used car? (circle)  Very  Somewhat  Not-so-good
   d. Find the fuel efficiency of any car? (circle)  Very  Somewhat  Not-so-good

2. Why do most cars get better fuel economy (higher mpg’s) on the highway than in the city? If you don’t know this answer, type the question into Google!

3. Why does a car dealer charge you more for the same used car than a private person? If you don’t know this answer, again—ask Google!

If you have time… a little history:

1. Go to: www.kbb.com/company/history

2. What was the full name of the man who the *Kelley Blue Book* is named for?

3. What state was he from? __________________________
   What year was he born? (You’ll need to do some MATH for this answer.) ______________________

4. Who was “Buster?” __________________________

5. What was Les very good at? What did he start to be known for? __________________________

6. What sport is mentioned throughout this article? __________________________

7. Why were all cars painted black in the 1920s? __________________________

8. When did the Kelleys quit the car sales business and start to only focus on the Blue Book? __________________________

9. When was *Kelley Blue Book* launched as the website www.kbb.com? __________________________

10. When kbb.com first started, people were charged for each car pricing report.
    How much were they charged? __________________________
    Why did kbb.com stop charging money? __________________________
Car Talk
Lesson 8

DIRECTIONS: You will have a little “car talk” with a teacher, parent, guardian, aunt, uncle, or any other adult. You need to find out the following information from the person you interview.

Your current vehicle: Year ______ Make ___________________ Model ___________________

Estimated miles you drive per month: ________________________________________________

Estimated $ you spend per year on maintenance: ______________________________________

A car you might consider buying: Year ______ Make ___________________ Model ____________

Would the car be purchased: □ New OR □ Used

Any other information that would influence your decision to change vehicles:
________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________

(For example: better gas mileage, need more room, current car isn’t cool enough…)
To Buy or Not to Buy a Car?
Lesson 9

Name: ________________________________
Date: ________________________________

Contents:
Overview and Directions
Exemplar Final Write-Up
Grading Assignment Rubric
Research and Costs
Summary and Conclusion
OVERVIEW: When it’s too far to bike, Ms. Ramos drives a 2005 Toyota Prius. But, just a few years ago, she was driving a 2003 Subaru Impreza. She did a lot of research and calculating to determine if it was worth it to trade in the Subaru and buy the Prius.

DIRECTIONS: The student is responsible to help someone make the same sort of decision. The student will evaluate the cost to own, drive, and maintain someone’s current car and compare that to the cost of buying, driving, and maintaining a newer car.

Your complete answer to this question will include

1. An equation for the cost to own, drive, maintain the current car for x years.
2. An equation for the cost to buy, drive, and maintain the newer car for x years.
3. A single data table that compares the cost of owning the two cars for x years.
4. A single graph that compares the cost of owning the two cars for x years.
5. A complete written explanation with
   ▶ An introduction, stating the purpose of the assignment
   ▶ Your findings and, ultimately, your recommendation. This should include clear and complete reasons for your recommendation—supported with data.

Car: 2003 Subaru Impreza
Gas Mileage: 26 MPG
Notes on Car:
- Current Car—no needed purchase
- Maintenance is approximately $1,000 per year
- Trade-in value is $3,500
- Gas only
- With 26 MPG, the 4,800 miles Ms. Ramos drives would cost $653 for gas

Car: 2005 Toyota Prius
Gas Mileage: 45 MPG
Notes on Car:
- MSRP $9,000
- 6.25% Sales Tax would be an additional $687.50
- No maintenance needed!
- Gas and electric
- With 45 MPG, the 4,800 miles Ms. Ramos drives would cost $362 per year for gas
EXEMPLAR FINAL WRITE-UP (not including graph and table)

Ms. Ramos was considering selling her 2003 Subaru Impreza (LEFT) and buying a 2005 Toyota Prius (RIGHT). Although the Prius was a little expensive, she thought that it was a lot more economical (it gets more mpg and won’t require any maintenance) so, eventually, it will be the better car for her budget. She calculated the upfront and yearly costs of each car; created equations; and then graphed the equations in order to determine if it was a good idea to buy the Prius.

She started by using the *Kelley Blue Book* (online) to find the mpg of each car, cost of the Prius, and trade-in value of the Subaru. The Prius would cost $9,000. When she included 6.25% tax on the car—the total cost is $9,562.50. But, she found out that the Subaru could be traded in for $3500. This would lower the cost of buying the new car to $6,062.50. The mpg of the two cars was very different. The Prius got 45 mpg; while her Subaru only got 25 mpg. That made a BIG difference in how much she’d spend on gas for the 4,800 miles she drive each year—$653 vs. $362! Not only would she spend more on gas for the Subaru, but, since it’s an older car it required about $1,000 in maintenance each year. Meanwhile, since the Prius was a pretty new car, she doesn’t believe it will require any maintenance.

*Here are her equations:*

New car (the Prius):  
\[ y = 6062 + 362x \]  
\[ y = 6100 + 400x \]  
(big upfront costs, but low yearly costs)

Current car (the Subaru):  
\[ y = 1653x \]  
\[ y = 1700x \]  
(no upfront costs, since I already own this car—but the yearly costs are very high.)

<table>
<thead>
<tr>
<th>Years of owning car</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cost of Current Car (Subaru)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Cost of Potential Car (Prius)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As she can see on the data table and the graph, it will only take between 4 and 5 years for the cost of the Subaru to catch up with the cost of buying the Prius. This is because the yearly costs of the Subaru are so much higher than the Prius ($1,700 vs. $400). It seems like the Prius is a good choice. Ms. Ramos should trade in the Subaru.
# To Buy or Not to Buy a Car? | Grading Rubric

<table>
<thead>
<tr>
<th>(10 pts) How effectively you use class time</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Staying on task</td>
</tr>
<tr>
<td>• Asking for help appropriately</td>
</tr>
<tr>
<td>• Efficient use of time</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(20 pts) Packet</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Neat</td>
</tr>
<tr>
<td>• Complete</td>
</tr>
<tr>
<td>• Accurate</td>
</tr>
<tr>
<td>• Units are included for ALL questions</td>
</tr>
<tr>
<td>• Work is shown where asked</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(20 pts) Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Ruler is used for all lines</td>
</tr>
<tr>
<td>• Title</td>
</tr>
<tr>
<td>• Units</td>
</tr>
<tr>
<td>• Labels</td>
</tr>
<tr>
<td>• Equations for both cars are included</td>
</tr>
<tr>
<td>• Names of both cars are included</td>
</tr>
<tr>
<td>• At least 10 years of data are included</td>
</tr>
<tr>
<td>• Overall—clear and easy to read</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(25 pts) Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Written in the 3 paragraph format described</td>
</tr>
<tr>
<td>• Completeness—includes the information described</td>
</tr>
<tr>
<td>• Organized and clear—information is easy to understand</td>
</tr>
<tr>
<td>• Recommendation is logical and clear</td>
</tr>
<tr>
<td>• Specific data is included and used to support the recommendation</td>
</tr>
<tr>
<td>• Spelling and grammar have few or no mistakes</td>
</tr>
<tr>
<td>• Typed or written clearly in ink</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(20 pts) Presentation (based on a complete answer to the question and the conclusion)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Integration of visuals</td>
</tr>
<tr>
<td>• Includes the academic vocabulary of linear modeling</td>
</tr>
<tr>
<td>• Vocal delivery skills—projection, articulation, rate and pauses</td>
</tr>
<tr>
<td>• Physical delivery skills—posture, gestures, eye contact</td>
</tr>
</tbody>
</table>

| (5 pts) On-Time |

| Total Points (out of 100) |
To Buy or Not to Buy a Car?
Lesson 9

DIRECTIONS: Complete the organizer and compare the cars.

RESEARCH AND COSTS

Introduction: Summarize the goal of this assignment.

The Cars:
1. Cars being compared:
   - Current Car: Year _______ Make _____________ Model _________________
   - Potential Car: Year _______ Make _____________ Model _________________

The Research:
2. Use *Kelley Blue Book* to find the following information:

<table>
<thead>
<tr>
<th></th>
<th>Current Car</th>
<th>Potential Car</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year, Make, Model</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MPG (miles per gallon)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&quot;fuel economy&quot;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSRP (purchase price)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trade-in Value</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Calculate Upfront Costs (these are one-time costs the need to be paid just to have the car):
3. Potential Car: Price __________________________
4. Sales Tax: In Massachusetts, we need to pay 6.25% sales tax on a car. Calculate the exact amount of sales tax that will be added to the purchase price of the potential car. Show your work.
To Buy or Not to Buy a Car?
Lesson 9

5. So, what is the total amount you'll have to pay (purchase price + sales tax) for the potential car?
   Show your work.

6. What is the trade-in value of current car?

7. Since the current car will be traded in, the money from the trade-in can be used to pay for the potential car. So, how much money will be needed up front or to start in order to buy the new car? (Use your answers to #5 and #6.) Show your work:

8. You just calculated the upfront costs for the potential car. What are the upfront costs for the current car that is already owned? Think: How much money is needed in order to own this car?

9. Now you have the one-time, up front costs of each car. Write them here:
   Current car: ___________________________
   Potential car: ___________________________

Calculate Yearly Costs (these costs need to be paid each year the car is used):

10. How many miles are driven each month?

11. So, how many miles does this come to each year? Show your work.

12. What is the fuel efficiency of each of the cars? Current car: _________ Potential car: _________
13. Use the mpg (miles per gallon) information for each car and the total miles driven each year to determine how many gallons of gas need to be purchased for each car for a whole year of driving. Show your work and circle the final answers.

<table>
<thead>
<tr>
<th>Current Car</th>
<th>Potential Car</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

14a. The cost of gasoline is currently about $1.80 per gallon. Use your previous answers to calculate how much money would be spent on gasoline for each car for an entire year. Show your work and circle the final answers.

<table>
<thead>
<tr>
<th>Current Car</th>
<th>Potential Car</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To Buy or Not to Buy a Car?
Lesson 9
### 14b. How much does each car cost to maintain each year?

<table>
<thead>
<tr>
<th>Current Car</th>
<th>Potential Car</th>
</tr>
</thead>
<tbody>
<tr>
<td>This is provided in the car information from the adult.</td>
<td>We will assume that since this is a newer car, it will NOT need maintenance.</td>
</tr>
</tbody>
</table>

### 15. So, what is the total amount of money that needs to be paid for each car every year?
To find this, you need to find the sum of the cost of gas and the cost of maintenance for each car. Calculate these. Show your work and circle the final answers.

<table>
<thead>
<tr>
<th>Current Car</th>
<th>Potential Car</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Making Equations:
You are going to be putting these upfront and yearly costs together to create a linear equation (in the form of $y = mx + b$) that will show the total cost ($y$) of owning each of the cars for $x$ number of years.

Which costs (upfront or yearly) will make up the $y$-intercept? ___________________________ Why?

Which costs (upfront or yearly) will make up the slope? ___________________________ Why?
To Buy or Not to Buy a Car?
Lesson 9

Current Car:
Write your equation: ____________________________
Re-write your equation with all values rounded to the nearest 100's: ____________________________

Potential Car:
Write your equation: ____________________________
Re-write your equation with all values rounded to the nearest 100's: ____________________________

Data Table: Create a data table that compares the cost of owning each of the cars for 10 years

<table>
<thead>
<tr>
<th>Years of owning car</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Car</td>
<td></td>
<td></td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Potential Car</td>
<td></td>
<td></td>
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<td></td>
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<td>(_____)</td>
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<td></td>
</tr>
</tbody>
</table>

Graph: Graph your equations on the same grid.
• Be sure to include: labels, title, units, key, and so on.
• Include the costs for years 0-10 (at least).
• Include the specific cars that are being compared.
• Include the equation for each car.
SUMMARY AND CONCLUSION:

**Paragraph 1: Goal**

- What was the purpose of the activity?
- Whose cars are the topic of your work?

Include the specific types of cars and what you hoped to find out by doing this work.

**Paragraph 2: Summarize (NOT EVERY DETAIL!)**

- What information did you use?
- Where did you find the information?
- How did you use the information?

**Paragraph 3: What were your findings?**

- Start by explaining why the two equations are so different.
- What does the graph show?
- What recommendation can you make to the adult who is considering trading-in their current car for a different car?

Be specific when you make your recommendation—refer to specific numbers of years and exact dollar amounts. You should be using specific data from your table and/or graph.
Exit Ticket and Feedback Form
Lesson 9

Exit Ticket: To Buy or Not to Buy a Car?  Name: ________________________________

Based on the feedback you received from your partner, how will you make improvements to your presentation? Please provide specific examples.

Feedback Form: To Buy or Not to Buy a Car?

Your Name: ________________________________  Presenter’s Name: ________________________________

Please describe one area of strength and one area where the presenter could improve.
You may refer to the content information in the presentation or the delivery skills of the presenter.

One strength:

One improvement:
Modeling Using Exponential Functions

TOPIC SEASON | Exponential Functions

This unit is designed for use in long-term programs. Sections may be adapted for short-term settings

Unit Designers J. Taylor, C. Huestis, K. Chace, R. Dubuisson, and J. Baer-Leighton

Introduction

We want our students to understand exponential growth and decay so that they have an understanding of how they see these things in their daily lives. After studying this unit, students will have a better understanding of how the human population grows or declines, how compound interest or depreciation in value works, and how bacteria can grow and disease can spread. Once students have an understanding of how exponential growth and decay occur, they can be aware of how their lives and the world can be affected immensely by changes that occur exponentially instead of linearly.

The “Modeling Using Exponential Functions” unit is designed as the third unit of Exponential Functions during the Winter Season and will take about two weeks to complete. Teachers in short-term settings will have the option of shortening the Performance Task and combining other lessons to complete this unit in six days. The unit asks students to explore:

Where in the real world do I see examples of exponential growth or decay?

How does learning about exponential functions help me understand real-world problems?

“After studying this unit, students will have a better understanding of how the human population grows or declines, how compound interest or depreciation in value works, and how bacteria can grow and disease can spread.”

These questions should frame a line of inquiry for students, allowing them to explore concepts of sustainability, as they will do in the Performance Task. In the Performance Task, students will be asked to find real-world examples of data sets that show exponential growth and decay and will be asked to show their findings in a presentation. The task illustrates the mathematical importance of providing data analysis, which is critical when making life or policy decisions.

If teachers want to expand this unit and make a broader connection to the study of science, teachers could give students scientific research and additional readings about the effect on the environment of exponential growth or decay such as population growth, the extinction of species, etc. Students could also research policy changes that have been put into effect because of the severe effects on the environment that exponential growth or decay can cause.

The “Modeling Using Exponential Functions” unit primarily focuses on the standards:

A-CED.1: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear, quadratic, and exponential functions with integer exponents.
F-IF.7.3: Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F-IF.8b: Use the properties of exponents to interpret expressions for exponential functions.

For example, identify percent rate of change in functions such as \( y = (1.02)^t \), \( y = (0.97)^t \), \( y = (1.01)^{12t} \), and \( y = (1.2)^{10t} \), and classify them as representing exponential growth or decay. Apply to financial situations such as identifying appreciation and depreciation rate for the value of a house or car some time after its initial purchase. (\( V_n = P(1 + r)^n \))

To understand the difference between exponential functions and linear functions, and to know what type of data fits an exponential model. The Performance Task requires students to apply their knowledge and skills from studying these standards to a real-world example of exponential growth or decay and to present their findings to the class through graphs, equations, data sets, and through a written explanation.

There are two important prerequisite skills that students need to be successful in this unit. First, students must know how to solve equations with one or two variables. Second, students should understand exponential rules. If students do not have these skills and need practice with these rules, a review lesson (Lesson 1) and pretest have been included in this guide. Teachers may choose to add additional practice days, if necessary.

For adaptation ideas for this unit, see p. 4.7.3 on the right.
### Exponential Functions: Modeling Using Exponential Functions

Adapting This Long-Term Unit for Short-Term Programs

#### Plan 1 (Long)

<table>
<thead>
<tr>
<th>WINTER SEASON—Modeling Using Exponential Functions: Long-Term Programs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MONDAY</strong></td>
</tr>
<tr>
<td>Week 1</td>
</tr>
<tr>
<td>Week 2</td>
</tr>
<tr>
<td>Week 3</td>
</tr>
</tbody>
</table>

#### Plan 2 (Short)

<table>
<thead>
<tr>
<th>WINTER SEASON—Modeling Using Exponential Functions: Short-Term Programs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MONDAY</strong></td>
</tr>
<tr>
<td>Week 1</td>
</tr>
<tr>
<td>Week 2</td>
</tr>
</tbody>
</table>

For a long-term program, the unit can stay as is.

For short-term program settings (Plan 2), teachers can shorten the Performance Task by providing students with predetermined data sets instead of asking students to find their own data sets on the website. Teachers can simply ask students to look at data sets, create an equation, and graph it. The short-term unit also condenses the introduction to exponential functions lessons into one day. The teacher could spend less time going over the PowerPoint in the original Lesson 3 and use the “Bugs, Bugs, Everywhere Bugs” from lesson 5 as a full-class lesson to instruct and allow students time to practice their skills. The short-term adaptation also eliminates practice time with exponential functions and graphs since students will be doing this in the Performance Task.

Lesson 1, “Exponent Rules” is a review lesson that can be added by the teacher to the beginning of either the short- or long-term plan if it is determined that it would be needed by the student(s). Activity Sheets are included in the Supplement.
UNIT PLAN

Long-Term Programs

Modeling Using Exponential Functions
Theme or Content Area: Algebra 1—Exponential Functions
Duration: 11 Lessons / 11-12 Days

Emphasized Standards (High School Level)

ALGEBRA

A-CED.1: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear, quadratic, and exponential functions with integer exponents.

FUNCTIONS

F-IF.7e: Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F-IF.8b: Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as \( y = (1.02)^t \), \( y = (0.97)^t \), \( y = (1.01)^{12t} \), and \( y = (1.2)^{t/10} \), and classify them as representing exponential growth or decay. Apply to financial situations such as identifying appreciation and depreciation rate for the value of a house or car some time after its initial purchase. \( V_n = P(1 + r)^n \)

Essential Questions (Open-ended questions that lead to deeper thinking and understanding)

Where in the real world do I see examples of exponential growth or decay?
How does learning about exponential functions help me understand real-world problems?

Transfer Goals (How will students apply their learning to other content and contexts?)

Students will conduct research, analyze data, organize information, and present their findings.

Students will apply their mathematical knowledge to determine and present a theory based on researched information.

Students will collect, organize, and model data in order to discover patterns, draw conclusions, and predict future outcomes.

For Empower Your Future Connections, see p. 4.9.1
### Learning and Language Objectives

**By the end of the unit:**

KUDs are essential components in planning units and lessons. They provide the standards-based targets for instruction and are linked to assessment.

<table>
<thead>
<tr>
<th>Students should know...</th>
<th>understand...</th>
<th>and be able to...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Functions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Domain</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exponents</td>
<td></td>
<td>Evaluate exponential functions for a given domain.</td>
</tr>
<tr>
<td>Powers</td>
<td></td>
<td>Evaluate the impact of exponential growth or decay in real-world situations.</td>
</tr>
<tr>
<td>Base</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exponential functions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exponential growth</td>
<td>Linear functions have a constant difference, whereas exponential functions have a constant ratio.</td>
<td>Identify and graph exponential functions.</td>
</tr>
<tr>
<td>Exponential decay</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant difference</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant ratio</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Growth and decay factor</td>
<td>Real-world situations can be represented symbolically and graphically.</td>
<td>Identify the growth or decay factor for a given exponential function.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Write exponential functions for a given situation and use them to solve problems.</td>
</tr>
</tbody>
</table>
Assessment Evidence

Quality questions raised and tasks
designed to meet the needs of all learners

Performance Task and Summative Assessment (see pp. 4.8.18-4.8.22)

Aligning with Massachusetts standards

Students will present data and the conclusions drawn from it. They must include a clear data set, identify independent and dependent variables, use multiple representations of the relationship within the data (i.e., words, table, equation, graph) and in all representations; show all of their math work in a neat and legible manner (e.g., calculations either written or typed); and create a presentation. The presentation may take any form, including PowerPoint, video, Prezi, poster, etc.

Students must create an introduction to the data they chose and explain why they chose it. They must also provide a conclusion stating what the data tells us about environmental sustainability. Students will use data provided on one of the following websites:

SEE: Statistic Brain
www.statisticbrain.com/

NASA Socioeconomic Data and Applications Center
sedac.ciesin.columbia.edu/search/data?contains=sustainability+data+sets

Quantitative Environmental Learning Project (QELP)
seattlecentral.edu/qelp

The QELP website offers a list of exponential scatterplots, including Earth’s Atmosphere Humidity vs. Temperature and Earth’s Atmosphere Pressure vs. Altitude.

SEE: Quantitative Environmental Learning Project (QELP)
seattlecentral.edu/qelp/Data_MathTopics.html#Exponential
Pre-Assessments (see pp. 4.8.9-4.8.10)
Discovering student prior knowledge and experience
- Prerequisite: Pre-Test, T-Chart
- Lesson 1: Review and Assessment

Formative Assessments (see pp. 4.8.12-4.8.21)
Monitoring student progress through the unit
- Lesson 3: Exit Ticket
- Lesson 4: Bugs, Bugs, Everywhere Bugs
- Lesson 5: Exponential Functions Scenarios and Graphs
- Lesson 5: Graphing Exponential Functions Activity Sheet
- Lesson 6: Exit Ticket
- Lesson 7: Data Table, Equation, and Graph
- Lesson 8: Exit Ticket
- Lesson 9: Exit Ticket
Access for All

Considering principles of Universal Design for Learning (UDL), Positive Youth Development/Culturally Responsive Practice (PYD/CRP), differentiation, technology integration, arts integration, and accommodations and modifications.

UDL is a framework for making curriculum more inclusive.

Multiple Means of Engagement

This is the why of learning. It is what makes students engage or disengage. Throughout the unit plan, the student will be provided with as many choices in the level of challenge and complexity as possible in order to recruit and sustain engagement. For example, the teacher will encourage and support students in setting their own personal, academic, and behavioral goals. The teacher will use many strategies to guide students, including reminders, guides, rubrics, checklists, and prompts among other things that focus students on self-regulatory goals. Student tasks will be varied, allowing for active participation, exploration, and experimentation. The teacher will provide differentiated models, scaffolds, and feedback, as well as content material that is culturally relevant and responsive to student’s backgrounds. Most important is that teachers design assignments and tasks with authentic outcomes, and that are purposeful and convey meaning to real audiences.

Students should be provided with as much choice as possible in the level of challenge, type of technology used (i.e. graphing calculators, Excel, and/or virtual manipulatives such as GeoGebra, InspireData, ExploreLearning, PhET Simulations and National Library of Virtual Manipulatives, and low-tech tools used such as graph paper, personal graphing whiteboards and a variety math manipulatives. The color, design, and layout of graphics, and sequencing and timing may also be adjusted to meet student needs. Students will be presented with concepts in various formats including but not limited to: literature, media-based resources, graphics and/or visual representations. Evaluative emphasis should be placed on process, effort, and improvement. Formative Assessments and Performance Tasks are designed to invite personal response, self-evaluation, and reflection. Students will have choice about what type of data to look at so that they are engaged in the Performance Task. Also, options for students to use internet-based quizzes and/or assessments as opposed to traditional worksheets will be provided. Students should be given problems and shown math examples that address a wide range of diversity and learning profiles in the classroom. Students will address real-world situations and applications. Where possible, assignments and brainstorming should be done collaboratively in pairs and/or small heterogeneous cooperative learning groups.

Multiple Means of Representation

This is the what of learning. There are many pathways to conveying information to students. Throughout the unit, the teacher will provide information and materials in several modalities such as diagrams, vocabulary cards, and word walls, posters, and charts with formulas for calculations; and models, videos, and audio for text. The teacher will also demonstrate concepts through hands-on activities.

The way information is displayed should vary, including size of text, images, graphs, tables or other visual content. Students are presented with information in multiple modalities and from a variety of sources (i.e., websites included in unit plan, Discovering Algebra text, and teacher generated materials, multi-media resources, visual graphics/representations). Where possible, written transcripts for videos and auditory content should be provided. Information should be chunked into smaller elements, and complexity of
questions can be adjusted based on prior knowledge competency. Reference sheets for examples, notes, vocabulary and definitions can be differentiated for content, process or product by differing the style of provided notes or providing pre-made flash cards or pre-formatted electronic versions for vocabulary and types of equations. By presenting information through a variety of culturally responsive means about exponents and functions of exponents the unit builds understanding in students that algebra has real-world applications and relevant meaning. Modifications intended to adjust the unit’s learning and language objectives, Transfer Goals, level of performance and/or content will be necessary for students with mandated specially designed instruction described in their Individualized Education Programs (IEPs).

Multiple Means of Action and Expression

This is the how of learning. In learning activities students will be provided options for demonstrating what they know and can do. Students will have access to assistive technology and use multiple media. For example, students will have access to word processors with grammar checks, word prediction, and spell checkers. Students could complete projects by making PowerPoint presentations, videos, storyboard, or drawing illustrations. In addition, students will have access to calculators. The teacher will scaffold writing or composing activities using tools such as concept maps, outlining tools, or graphic organizers. Students may need sentence starters and story webs to complete writing or composing tasks. The teacher will also break down long-term goals into short-term reachable goals.

Performance Tasks can be differentiated by content, process or product to address various learner profiles. Students may be given the choice to create short presentations using various tools including: Internet research, ThingLink, PowerPoint, Photostory, Excel, and Prezi to express their mastery of the learning and language objectives. Students should be given high- and low-tech options to compose in multiple media such as text, speech, drawing, illustration, comics, storyboards, design, film, music, visual art, sculpture, or video. All Formative Assessments will be provided in multiple forms including: hard copy, online quizzes, and student driven activities. Students can use graphic organizers, such as KTL Webs, two-column notes, concept mapping with Inspiration software, drawings by hand, checklists, sticky notes, reference sheets and mnemonic strategies, to better understand, brainstorm and demonstrate comprehension of the material. Students can also create typed flash cards or use a flash card app on an electronic device. Opportunities for peer collaboration, Think/Pair/Share, close reading, and whole-class discussion should be provided as needed. Accommodations intended to enhance learning abilities, provide access to the general curriculum, and provide opportunities to demonstrate knowledge and skills on all Performance Tasks will be necessary for students with applicable Individualized Education Programs (IEPs) and could benefit all learners.
Literacy and Numeracy
Across Content Areas

Reading
Students will read and follow the instructions and information for the assignments. Students will also read about data sets and read about how exponential growth and decay affect people and the world.

Writing
Students will engage in writing activities through Do Now activities or Exit Tickets. They will need to write to explain their reasoning and prove their answers through writing. For the Performance Task, students will be explaining why they selected the data set that they did and will write a concluding paragraph about what the data tells us about sustainability.

Speaking and Listening
Students will speak with their teacher and classmates in order to complete all of the assignments in this unit. During the review and assessment parts of lessons, students should share their reasoning with their classmates and build on the ideas of their classmates to clarify their own thinking. For the Performance Task, students will create a presentation that they will present to the class.

Language
Students will use content-specific mathematical vocabulary to explain the concepts discussed in this unit.

Numeracy
Students will work with data sets in order to understand exponential growth and decay. They will write equations and graph the equations.

Resources (in order of appearance by type)

Print

Websites
Lesson 1

Lesson 1
Lesson 1:
www.youtube.com/watch?v=CK8RYtBx-c4

Lesson 2
www.youtube.com/watch?v=GGytywqpGXA

https://sustainabilityissues.wordpress.com/about/

Lesson 8
www.learner.org/interactives/dailymath/population.html

Lesson 9
“Rubistar,” 4Teachers. ALTEC at University of Kansas. 2008.
http://rubistar.4teachers.org/

http://seattlecentral.edu/qelp/

www.statisticbrain.com/

“Socioeconomic Data and Applications Center.” Earth Data. The Trustees of Columbia University in the City of NY.

Materials (These are found in the Supplement unless otherwise noted.)

<table>
<thead>
<tr>
<th>Lesson 1:</th>
<th>Pre-Assessment on Exponent Rules (Quiz)</th>
<th>Activity Sheet</th>
<th>pp 4.10.1-4.10.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lesson 1:</td>
<td>Laws of Exponential Functions: Easy (Extension)</td>
<td>Activity Sheet</td>
<td>pp 4.10.3</td>
</tr>
<tr>
<td>Lesson 1:</td>
<td>Laws of Exponential Functions: Medium (Extension)</td>
<td>Activity Sheet</td>
<td>pp 4.10.4</td>
</tr>
<tr>
<td>Lesson 1:</td>
<td>ANSWER KEY—Lesson 1 Activities</td>
<td>Answer Key</td>
<td>p. 4.10.5</td>
</tr>
<tr>
<td>Lesson 3:</td>
<td>Exponential Functions Powerpoint <a href="http://bit.ly/2qp5Rvb">http://bit.ly/2qp5Rvb</a></td>
<td>PowerPoint</td>
<td>Google Drive (DYS/SEIS educators only)</td>
</tr>
<tr>
<td>Lesson 5:</td>
<td>Exponential Function Organizer</td>
<td>Activity Sheet</td>
<td>p. 4.10.6</td>
</tr>
<tr>
<td>Lesson 5:</td>
<td>Why Exponential Functions Matter (3 Scenarios)</td>
<td>Activity Sheet</td>
<td>p. 4.10.7</td>
</tr>
<tr>
<td>Lesson 5:</td>
<td>ANSWER KEY—Exponential Functions (3 Scenarios)</td>
<td>Answer Key</td>
<td>p. 4.10.8</td>
</tr>
<tr>
<td>Lesson 6:</td>
<td>Graphing Exponential Functions</td>
<td>Activity Sheet</td>
<td>pp. 4.10.9-4.10.12</td>
</tr>
<tr>
<td>Lesson 7:</td>
<td>Solving Exponential Equations PowerPoint <a href="http://bit.ly/2qp5Rvb">http://bit.ly/2qp5Rvb</a></td>
<td>PowerPoint</td>
<td>Google Drive (DYS/SEIS Educators only)</td>
</tr>
<tr>
<td>Lessons 9-11:</td>
<td>Sustainability Project Packet (student-designed)</td>
<td>Project Packet</td>
<td>pp 4.10.13-4.10.20</td>
</tr>
</tbody>
</table>
PREREQUISITES: Math skills needed for this unit

If the teacher feels students need to review exponential rules before beginning this unit, Lesson 1 will provide a content review and additional practice. If the teacher feels additional practice is unneeded, this unit can begin with Introductory Lesson 2.

Students should know:

- How to solve equations with one and two variables

  If students need to review exponential rules before beginning this unit, Lesson 1 will provide a content review and additional practice.

Outline of Lessons

Introductory, Instructional, and Culminating tasks and activities to support achievement of learning objectives

INTRODUCTORY LESSONS

Stimulate interest, assess prior knowledge, connect to new information

Note: Teachers should vet all videos included in this unit according to program standards and create templates or graphic organizers for students to monitor their comprehension of the material.

Lesson 1

Exponent Rules (Review of Prerequisite Skills)

Lesson 1 is designed as a review of prerequisite skills needed for student success, in preparation for introductory Lesson 2, “Exploring Growth, Decay, and Exponential Functions.”

Do Now (time: 15 minutes)

The teacher will start the lesson by helping students to activate their prior knowledge about simplifying expressions with exponents. As a class, students will define essential vocabulary that will appear in the pretest below. The teacher should ask the students to explain what they know about simplifying expressions with exponents.

Before giving the pretest on simplifying expressions with exponents, the teacher will explain what the Pre-Assessment is for and how it will help prepare students for learning about exponential functions.

The following website could be used, or use the Pre-Assessment Activity Sheets on pp. 4.10.1 to 4.10.4 in the Supplement. The ANSWER KEY for the Lesson 1 activities is on p. 4.10.5 in the Supplement. Grading can either be done by the teacher or by student partners, using a key posted by the teacher.
Hook (time: 10 minutes)
Show a long string of the same number being multiplied (e.g., $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$) and ask students what would be an easier way to write it ($5^{10}$). Do the same with 2 different digits (e.g., $5 \cdot 5 \cdot 7 \cdot 7 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 5$) and how it could be written with exponents ($5^6 \cdot 7^5$). Exponents are useful!

Presentation (time: 10 minutes)
The teacher should direct students to the following website, which demonstrates how to apply the basic rules of exponents. The teacher will review the rules of exponents found on the website. The teacher should provide supporting documents such as symbols charts, vocabulary and definitions, and/or a note-taking sheet to support student understanding. Students may also explore the website to discover the rules of exponents on their own with the guidance of these supporting documents.

Practice and Application (time: 15 minutes)
While the teacher is explaining the rules of exponents or while students are exploring the website, students should take notes on the rules. The students can use a T-chart to write the rules on one side and an explanation or example on the other.

Review and Assessment (time: 5 minutes)
The teacher will show students the five minute video on YouTube, “The MADSPM Jam,” to help students remember the rules of exponents.

As students watch the video, they will write down the rules for MADSPM. To do this, the teacher will give students a piece of paper. Students will fold their papers into thirds. They will write MA in one column, DS in another, and PM in the last. In each column, they will explain what each pair of letters means, providing an example from the video.

Extension
Students will create their own songs, poems, or mnemonics that describe the rules. There are additional math songs on exponents the teacher can show from YouTube.

Note: Teachers may also use the easy and/or medium versions of the “Laws of Exponential Functions” Activity Sheets on pp. 4.10.3 and 4.10.4 in the Supplement (see p. 4.10.5 for the ANSWER KEY).
Lesson 2
Exploring Growth, Decay, and Exponential Functions

Goal
Students will explain how exponential functions are used in a real-world situation.

Do Now (time: 10 minutes)
To introduce students to the idea of exponential functions, the teacher will show the video from Mashup Math. This imaginative video provides students with an example of how we can use exponential functions to determine how long it would take for the US population to become zombies.

SEE: How do I graph an Exponential Function? ...with Zombies!!!
www.youtube.com/watch?v=GGytywpqGXA

Hook (time: 10 minutes)
The teacher will have students complete a think/pair/share activity to discuss the following questions:

Why was the zombie population able to grow so quickly?
Why did the human population decline so quickly?

Once students can answer these questions about the situation presented in the video, the teacher will ask students to think about how this same concept affects us in real-life situations. Students can write their responses or share their responses orally.

What additional factors can influence population growth and why? Students might note that contagious diseases (Ebola, for example) can spread exponentially in much the same way if we are unable to control the spread of the disease. The teacher will ask students:

Using what we know about how real disease (or zombies!) can spread, how should governments plan to stop viral outbreaks?

(If the teacher would like to extend the hook beyond the suggested 10 minutes, the teacher could ask students to practice writing theories in a formal way and to use cause-and-effect and sequencing language. The teacher will provide students with sentence starters, if needed, to help students with cause and effect language. Sentence starters could look like the following:

Because diseases can spread exponentially if they are uncontrolled …
The government should … because diseases …

Presentation (time: 5 minutes)
The teacher will ask students to brainstorm other real-world situations where they could use the growth and decay models. (Examples include growth of bacteria, population growth, bank accounts accruing interest.) Where else could they use exponential functions to represent growth and decay?

The teacher should record the students’ ideas on the board. Students might come up with examples that are not real exponential growth or decay models, but this is fine for now. When they research in the practice and application section of the lesson, they will find more examples of real exponential growth and decay.
Practice and Application (time: 20 minutes)
Students will use computers to research examples of growth and decay that they found in the real world. If examples that they found aren’t examples of exponential growth or decay, students should realize it when they begin to research the examples. If students are having trouble thinking of examples to research, the teacher can make suggestions to students. For example, students might look up animal species extinction. Students can research animal populations numbers from 20 years ago, 10 years ago, and now and make predictions about extinction dates.

If students do not have access to computers and cannot complete this activity, the teacher can take the examples that students brainstormed and look them up with the class, projecting the data that he or she finds. The teacher will show students data on their topics and ask students to think about whether or not they are real examples of exponential growth or decay. Students can discuss each topic with a partner. The teacher can look up as many examples as he or she can in the time allotted, ensuring that students are discussing each topic and thinking about them in terms of exponential growth and decay.

Another option would be for the teacher to create a list of examples, some that are true examples of exponential growth and decay and some that are not. The teacher could ask students to label them as TRUE examples of FALSE examples after giving students time to think about the examples and discuss them with a partner.

Review and Assessment (time: 10 minutes)
Students will share their examples with the class and explain why the examples that they found are good examples of exponential growth or decay.

INSTRUCTIONAL LESSONS
Build upon background knowledge, make meaning of content, incorporate ongoing Formative Assessments

Lesson 3
Introduction to Exponential Functions 1

Goal
Students will be able to explain the concepts of exponential functions and will use appropriate vocabulary when explaining the concepts.

Do Now (time: 10 minutes)
The teacher will ask the students to work in pairs to decide if they would rather have $10,000 right now or opt for the “Penny Plan” where they would only get 2 pennies on the first day, but have the amount of pennies double every day for 20 days.

Note: After 20 days, a person on the “penny plan” would have $20,971.50.
Students should write down their explanations and work out the math to support their answers. Students may use manipulatives such as paper slips, tick marks on a paper, or coloring boxes on chart paper to support their computations.
Hook (time: 10 minutes)
Students will present their responses and their mathematical evidence to the class. When the students are presenting their answers to the class, the teacher will ask the students questions such as:

- How did the pattern of data change on the first few days compared to the last few days?
- What did you notice about how much money the person on the “Penny Plan” was making during the first week of the plan?
- What about how much money they were making the last few days of the plan?

Students should begin to see how quickly growth happens when it occurs exponentially.

Presentation, Practice, and Application (time: 25 minutes)
The teacher will use the “Introducing Exponential Functions” PowerPoint to present the lesson on Exponential Functions, available in the Google Drive Math Guide resource for DYS/SEIS educators. The teacher will refer to the Zombie apocalypse video from the day before to connect ideas in the students’ minds. While the teacher is presenting, s/he should stop periodically to allow students to ask questions and take notes. The teacher can instruct students to draw pictures with their notes to ensure that they have visuals to aid in their understanding. There are also practice examples within the PowerPoint presentation that students should use to practice as the lesson goes on. This reinforces some of the concepts within the lesson.

SEE: Exponential Functions PowerPoint

Review and Assessment (time: 10 minutes)
The last slide of the PowerPoint includes questions for the students to answer. The teacher can ask students to respond verbally to the questions or can ask students to write down their answers as an Exit Ticket to turn in before they leave for the day.

### Lesson 4

Introduction to Exponential Functions 2

Goal
Students will explain the difference between linear functions and exponential functions.

Do Now (time: 10 minutes)
The teacher will write the following problem on the board and ask students to solve:

Selena’s starting salary for her new job is $32,000 per year. She calculates her projected salary for the next 5 years by using the function $Y = 32,000(1.12)^x$

Students will identify the initial value, the growth or decay factor, and the growth or decay rate of the exponential function and answer the question:

- What will be her salary after 3 years, 5 years?
Hook (time: 10 minutes)
Before sharing their responses and mathematical approaches with the class, students will self-assess their understanding and their processes. The teacher will ask students to rate themselves on their understanding and the efficiency of their process. Students will then share their responses and mathematical approaches with the rest of the class as they build understanding and competency together. As a final step, students will reflect on their own self-assessment after they have shared their responses to identify areas of success or areas for improvement.

Practice and Application (time: 25 minutes)
The teacher will walk students through steps 1-8 of the activity “Bugs, Bugs, Everywhere Bugs” on p. 333 in Discovering Algebra: An Investigative Approach, 2007 EDN. See the Teacher’s Guide (p. 367) for detailed explanations.

Review and Assessment (time: 10 minutes)
The teacher will set up a think/pair/share activity and have students think about the following questions:

- How does the population growth of the bugs differ from linear growth?
- What can we say about the differences between linear functions and exponential functions?

Note: A linear function grows (or shrinks) at a constant rate called its slope. An exponential function grows (or shrinks) at a rate which increases (or decreases) over time. From a practical standpoint linear growth (or shrinkage) is simple and predictable. Exponential growth is essentially out of control and unsustainable and exponential decay soon becomes negligible.

Lesson 5
Why Exponential Functions Matter

Goal
Students will explain how exponential functions can be used in real-world situations.

Do Now (time: 10 minutes)
Students will use the “Exponential Functions Organizer” Activity Sheet (p. 4.10.6 in the Supplement) with partners as a way to begin the conversation about sustainability and exponential functions.

The teacher should let students wrestle with the questions and ask each pair to respond to one of the three prompts. Students should make a few notes about the question they selected so that they can share their thoughts with the class.

Hook (time: 10 minutes)
Each pair of students will respond to at least one of the prompts on the Activity Sheet. Allow students time to explain their thinking. The teacher will then ask the whole group:

Why should exponential functions matter to us?
The teacher should take some notes on the board as the students are discussing this question.

**Presentation** (time: 5 minutes)
The teacher will tell students that exponential functions matter a lot in our daily lives. The teacher will tell students that they are about to study three scenarios, one at a time, to explore why exponential functions matter. The three scenarios can be found in the “Why Exponential Functions Matter” Activity Sheet on pp. 4.10.7 of the Supplement. The teacher will hand out the first scenario to each pair of students. As pairs finish the first scenario, they will be given the second scenario. When they finish the second scenario, they will be given the last scenario.

**Practice and Application** (time: 20 minutes)
Students will work in pairs on each scenario. As the students are working on the scenarios, the teacher should circulate around the room and help students as needed. If the teacher does not think that there is enough time for the students to complete all three scenarios, the teacher might decide to give each pair of students a different scenario and ask the pairs to share the scenario they worked on during the review and assessment. The teacher might decide to extend the lesson into the next day and allow students more time to practice with these scenarios.

**Review and Assessment** (time: 10 minutes)
The teacher will walk through each scenario result (see Why Exponential Functions Matter Scenario Results Activity Sheet on p. 4.10.8) with students, allowing a pair to present their solution and approach before debriefing with the class. It’s important for students not only to present answers to the questions, but to show the data tables and graphs they’ve created.

**Lesson 6**

Graphing Exponential Functions

**Goal**
Students will explore graphs of exponential functions.

**Do Now** (time: 15 minutes)
The teacher will provide students with graph paper and ask them to work in pairs to graph the following: 
\[ Y = 2^x, \quad Y = 3^x, \quad Y = 4^x \]

The teacher will then ask students to use the same values for \( Y \) in each graph (-2, -1, 0, 1, 2, 3) and to find the corresponding \( x \) values that make the points on the graph.

Students should be prepared to present their solutions and observations to the class.

**Hook** (time: 10 minutes)
When students finish creating their graphs, the teacher will remind students that they’ve seen some of this done already with the Zombie video that they watched the first day of the unit.

In pairs, students will present their solutions and observations to the class.
The teacher can post the graphs on the board so that students can see all of them at the same time. The teacher will ask follow up questions such as:

What do you notice about the graphs?
Is there any pattern?
Based on what we observe in the graphs, can we create a rule about graphs of exponential functions?

For example, students should note that you can’t raise a positive number to any power and get 0 or get a negative number. To guide students, the teacher will ask questions such as:

Is there ever any value of \( x \) that will make \( y = 0 \)?
Is there ever any value of \( x \) that will make \( y \) negative?

**Practice and Application** (time: 20 minutes)
Students will continue to work in pairs to complete the “Graphing Exponential Functions” activity on pp. 4.10.9-4.10.12 in the Supplement. The teacher should circulate around the room to help students as needed.

**Review and Assessment** (time: 10 minutes)
In pairs, students will select one problem that they’d like to share with the class. They must share the problem, their understanding of it, and their approach to solving it. After students have shared their problem and their approach, they will complete a self-reflection about how effectively they collaborated, communicated, and solved the problem. Students will identify ways they can improve these skills and make these types of activities more effective and efficient.

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**Lesson 7**

**Solving Exponential Equations**

**Goal**
Students will use the equality property of exponential functions to solve exponential equations.

**Do Now** (time: 10 minutes)
The teacher will ask students to create two graphs: one that shows exponential growth and one that shows exponential decay. Students will write the equations for their graphs on the back of their graphs so that they can verify their classmates answers in the next step of the lesson plan.

**Hook** (time: 10 minutes)
The teacher will ask students to present their graphs. (If the teacher has access to a document projector, students should use this to project their graphs.) As each student shows his or her graph, the teacher will ask the rest of the class to determine the equation for that graph. The teacher can also ask students to identify the rate of growth or the rate of decay depending on the graph.

**Presentation, Practice, and Application** (time: 30 minutes)
The teacher will show the Solving Exponential Equations PowerPoint (located on Google Drive) to
the class. The teacher should work his way slowly through the slides, stopping in case students have questions and allowing them time to take notes. There are also practice examples within the PowerPoint presentation that students should use to practice as the lesson goes on. This reinforces some of the concepts within the lesson.

**SEE:** Solving Exponential Equations PowerPoint

### Review and Assessment (time: 5 minutes)

The teacher will ask students to respond to summary questions included in the PowerPoint presentation. Students can respond to these questions verbally or can write down their responses to turn in as an Exit Ticket.

### Lesson 8

#### Review and Practice with Exponential Functions

**Goal**

Students will review and practice exponential functions skills that they have learned so far in this unit.

**Do Now (time: 10 minutes)**

The teacher will have students visit the following website:

**SEE:** Math in Daily Life: Population Growth
www.learner.org/interactives/dailymath/population.html

The teacher will allow students time to read the text and think about responses to some of the questions posed. The teacher will likely need to provide guidance documents, a handout to complete, or a note-taking sheet (Cornell notes or two-column notes) to support independent analysis and discovery.

If students cannot visit the site on their own, the teacher can print out the “Population Growth” and “Getting the Picture: Communicating Data Visually” pages. The teacher will ask students to think about where else in the world or in real life they see examples of exponential functions.

**Hook (time: 10 minutes)**

The teacher will ask students to work in pairs to come up with a few real-world examples of exponential functions If students are having a difficult time thinking of topics, the teacher will ask them to think about categories such as population or finance. The teacher will prompt students who are having difficulty with topics, or will share some ideas with the class (bacteria growth, compound interest, depreciation of car values, and so on). Students should think of at least three examples.

The teacher will ask students to answer the following questions about their examples:

- Where in the real world do you see these examples?
- Do the examples reflect growth, decay, or can it be both?
- What data would you need to collect to test and graph a function about your example?
- Where could you collect this data?
Presentation (time: 5 minutes)
The teacher will tell students that they will work in pairs to select one example, find the data, and create the following:

- A sample data table to show the data they’ve collected
- An equation to show a relationship in the data
- A graph expressing that relationship
- Identification of the growth or decay rate and the growth or decay factor

Practice and Application (time: 20 minutes)
Students continue working in pairs to find the needed information and to create a data table, equation, graphs, and identify growth and decay rate and factor. The teacher may need to assist students in finding the necessary information. If students are not able to find this information by accessing the internet independently, or if internet access is limited, the teacher will provide students with the information. The teacher will have a few options available for the students to choose from.

Review and Assessment (time: 10 minutes)
In pairs, students will present their findings to the class. The teacher should ask students questions as they present to ensure that they have a good understanding of what they found. The teacher will ask students:

- How did you determine the decay or growth rate?
- Where did you find the data that you used in your table?
- How did you decide how to express that data in an equation?

CULMINATING LESSONS

Includes the Performance Task, i.e., Summative Assessment—measuring the achievement of learning objectives

Lesson 9

Sustainability Project—Preparation (2 days)

Students will spend two days creating and finalizing a Sustainability Project Performance Task to assess student understanding of Exponential Functions. The project will culminate in class presentations followed by a feedback exercise.

Students will research, analyze data, and prepare a written and visual presentation for this assignment:

Students will spend two days researching, analyzing data, and preparing a written and visual presentation for the following assignment:

Imagine that your class has received a request from the non-profit organization, Learning for a Sustainable Future (LSF). As a part of their classroom outreach, LSF is asking students to research one topic related to environmental sustainability and prepare a report that explains their findings and makes predictions or draws conclusions about environmental sustainability. Each student should research a topic of their choosing, find an exponential data set, then create
multiple representations of the data, analyze the data, and prepare a presentation that discusses the implications of the data with regard to sustainability. After the students receive peer feedback on their presentations, they will make adjustments before presenting to the entire class and passing in their final report.

**Goal:** Students will use exponential data to explain real-world environmental sustainability challenges.

**Role:** You are a student-researcher involved in an outreach program conducted by Learning for a Sustainable Future (LSF), a leading Canadian non-profit organization that was created to increase sustainability education. The organization asked you to study a topic related to environmental sustainability, collect data, and report on your findings.

**Audience:** The non-profit organization, Learning for a Sustainable Future, is interested in the results of your research and data analysis on an environmental sustainability topic. Your classmates will listen carefully to your presentation and provide feedback before you finalize your report for LSF.

**Situation:** In 2004, the United Nations Educational, Scientific, and Cultural Organization (UNESCO) declared The United Nations Decade of Education for Sustainable Development (ESD) from 2005–2014. This effort brought world focus to the important issues of education for sustainable development. As a continuation of this effort, many international organizations are providing opportunities for classroom research on these vital issues.

**Product:** A sustainability report and presentation that will analyze and interpret the data gathered on one topic related to environmental sustainability.

**Standards:** A clear data set on one topic related to environmental sustainability, identification of the independent and dependent variables, multiple representations of the relationship within the data (i.e. words, a table, an equation, and a graph), explanations of all representations, displaying all math work in a neat and legible manner (e.g., calculations), and preparing an informative presentation (such as a video, PowerPoint, Prezi, or poster).

**Lesson 9—DAY 1:**

**Goal**
Students will select an environmental sustainability topic for their exponential functions Performance Task and understand the data set that they are working with.

**Do Now** (time: 10 minutes)
The teacher will distribute the “Sustainability Project” Project Packet for the Performance Task (pp. 4.10.13-4.10.20 in the Supplement) and allow students to read it aloud together. After reading the document, students will write any questions that they have on the document. If something needs to be clarified, they will write down those questions, too.
**Hook** (time: 15 minutes)
The teacher will ask students:

What do you need to know and be able to do in order to complete this task successfully?
What are some ways you can present your information?

Students will brainstorm a list of the information that they need to compile and a list of ways that they can present their findings to the class. Students might want to create a video, a PowerPoint, a Prezi, a poster, etc.

Once students have a list of what they need to find to complete the project and how they might present it, the teacher and students should work together to create a rubric for the Performance Task. Students should use the “Sustainability Project” Project Packet (pp. 4.10.13-4.10.20 in the Supplement) and the brainstorming list to create the criteria for the rubric. The teacher might use a website like Rubistar to help create the rubric.

**Note:** There is list of content assessment criteria included in the process steps section of the “Sustainability Project” materials in the Supplement. The teacher can also include these criteria in the rubric developed by the class. The teacher should carefully review the rubric with students.

**Presentation** (time: 30 minutes)
The teacher should project one of the following websites:

**SEE:** Statistic Brain
www.statisticbrain.com/
Socioeconomic Data and Applications Center
sedac.ciesin.columbia.edu/search/data?contains=sustainability+data+setse
Quantitative Environmental Learning Project
seattlecentral.edu/qelp/

The teacher will guide students to the various data sets that exist on the site and show students where they can find the information that they need. Students can select any topic that is of interest to them, so the teacher should show students how many data sets exist on the sites. The teacher will select a sample that is of interest to him/her and do an example with the students. The teacher could use any of the exponential sets listed on the QELP website here:

**SEE:** QELP
seattlecentral.edu/qelp/Data_MathTopics.html#Exponential

The teacher will select a topic, review the data available, and talk through how students could select the data and create a presentation of exponential growth or decay using that set of data.

**Note:** If the teacher chooses to use data from the Socioeconomic Data and Applications Center the user will need to create a log-in ID and password with NASA Earth. The website will provide instructions and there is no cost involved.

This presentation might take students to the end of DAY 1. (The timing will depend on how many questions the students ask while the teacher is modeling for students how to put together the Performance
Lesson 9—DAY 2:

Practice and Application (time: 50 minutes)
The teacher will give students time to begin working on their project. By the end of today’s class, students should have selected a topic and have an understanding of the topic and data set. Students might need more time than the time allotted here to search for a topic of interest to them that also shows exponential growth or decay. The teacher should give students as much time as needed for them to find a topic that interests them and to understand the data set on the website.

Students can use the Quantitative Environmental Learning Project (QELP) website, Statistic Brain, or the Socioeconomic Data and Applications Center websites (see links provided in the Presentation) to find data sets that they want to explore for the final sustainability project. If students do not have free access to the computer, the teacher will show students the various data sets available, make a list of the topics that interest students, and print out the information that students need to complete the project.

Review and Assessment (time: 5 minutes)
As an Exit Ticket, students will write down their chosen topic and what data they will explore and present in their work. The students should also write down how they think they will present the data that they are working with.

Lesson 10

Sustainability Project—Workplan Creation

Goal
Students will create their Sustainability Project Workplan.

Do Now (time: 5 minutes)
The teacher will distribute index cards to students, and ask them to take five minutes to create an agenda or a plan for their work for this period.

Note: The teacher might have to show students an example of a work plan if students are unsure of how to create one.

The teacher will tell students that this is how professionals work; they set a plan for themselves to ensure that they are using their time wisely and completing all of the tasks that they need to complete. The teacher will tell students that they will review the plan at the end of the period to see if they were successful. Process steps are contained in the “Sustainability Project” Project Packet on pp. 4.10.13-4.10.20 of the Supplement.

Hook (time: 5 minutes)
The teacher will hand out a copy of the rubric that the students created yesterday and will review the expectations for the Performance Task. Students should ask any clarifying questions that they have and
should compare the rubric’s expectations to the plan that they made for themselves today.

Did they forget to include anything on their plan that must be in their final project?

Students can revise their plan if needed.

**Practice and Application** (time: 40 minutes)
As students are working on their tasks, the teacher is walking around to answer questions and make sure students are making progress and using their time productively.

**Review and Assessment** (time: 5 minutes)
Have students review the work plan they created on the index card at the beginning of class and cross out what they actually did and circle what they have left to do. The teacher should ask students:

Were you able to accomplish your goal today? Why or why not?

Students should write their answers on the back of their index cards and submit them to the teacher. The teacher can look through the index cards and decide how much more time students need to complete their Performance Tasks.

### Lesson 11

**Sustainability Project—Presentations**

**Goal**
Students will finalize their presentations and present their information to the class.

**Do Now** (time: 5 minutes)
The teacher will hand back the Exit Ticket index cards from yesterday’s class and students will review what they need to accomplish today.

**Hook** (time: 5 minutes)
Students will compare what they have already completed for their Performance Tasks to the rubric. How would they rate their projects so far? What might they need to improve today?

**Practice and Application** (time: 20 minutes)
Students finalize their Performance Tasks. The teacher should circulate around the room to help students as needed.

**Review and Assessment** (time: 25 minutes)
Students will practice their presentations with a partner. The partners should give each other feedback on the presentation before the students present to the entire class. The feedback can be in the form of a 3, 2, 1. The partner can say three things that the student did well in the presentation, two pieces of information that he/she found interesting, and one thing that the student can improve before presenting to the class. Students can also use the Performance Task rubric developed in class or the Future Ready Rubric to self-assess and to provide feedback to each other.

Each student presents their topics, data sets, math work, and conclusions to the class.
Connections to Empower Your Future
UNIT: Modeling Using Exponential Functions

Future Ready Connections

This unit encourages students to practice their Future Ready skills, think creatively and critically, and to reflect on their computational methods and results. Throughout the unit, students are encouraged to experiment, share their discoveries, reflect, and try new approaches to solving exponential functions problems and using math tools.

_This freedom to experiment will give teachers the opportunity to evaluate students on their initiative, self-direction, and accountability_ for completing tasks, actively engaging in critical thinking, and taking responsibility for their own discoveries. Teachers should reflect on whether or not youth stay on task without prompting, work together respectfully and cooperatively, and if they push themselves to thoroughly complete each activity, answer their own questions, and create a detailed final product instead of only addressing the minimum required information or waiting for explanations from the teacher. Youth have many opportunities to strengthen their communication and listening skills through group discussions and the presentation of their Performance Tasks. Students can also be asked to communicate through writing which can be evaluated for clarity and effectiveness.

_Teachers are encouraged to use the Future Ready Rubric_ to evaluate students’ growth and are encouraged to have students self-evaluate their progress using the Future Ready Rubric.

Transfer Goal Connections

The Transfer Goals in this unit emphasize students’ ability to conduct research, formulate a theory or draw conclusions, and present the information effectively and clearly. These Transfer Goals connect with Possible Selves elements which are the foundation of the Empower Your Future curriculum. The Possible Selves element that most directly connects with this unit is “Learn How to Learn.”

As students progress through the unit they will need to explore how they best learn and be able to make changes to their processes and methods. They will need to reflect on what allows them to be a successful student and lifelong learner. Teachers should consider expanding on these Transfer Goals by having youth reflect on the process of researching, problem solving, and the need to think “outside the box.” Teachers may ask students to reflect on their own learning and discovery process when they come across a new task or challenging situation.

What do they do first?
What if their initial attempt doesn’t work?
How do they decide to try something new?

Teachers can make this connection for students by encouraging students to reflect on how they conducted their research and which steps of the process needed to be modified or required support. Students can also track and reflect on how they formulated a theory or came to a conclusion so that they can apply this process to future tasks and decision-making situations.
“Teachers should encourage students to see that knowledge of exponential functions does play a role outside of the classroom and in many career fields.”

PYD/CRP Connections

This unit reflects many aspects of Culturally Responsive Practice and Positive Youth Development that empower students to be experts in the classroom and active leaders in their education. The unit utilizes humorous and interesting examples of growth and decay (such as the zombie invasion video in Lesson 2) that will hook students and engage them in the content. The lessons frequently allow for youth to have a voice as they choose research topics, determine their methods for recording data, and decide how to present the information to their peers (Performance Task, Lesson 4 and 5 Hook, Lesson 6 Review and Assessment). By encouraging youth to take on leadership roles and to be part of the discovery process, teachers are grounding the tasks in the students’ strengths and allowing them to develop not only their mathematical skills but their Future Ready skills (accountability, communication, cooperation, etc.) that will prepare them for the workforce. The unit also encourages students to identify other real-world examples of exponential functions and graphs in their daily, academic, or future professional lives (Lesson 8). This helps youth create meaningful connections to the topic and deepen their understanding of how this knowledge appears in the world. The lessons allow for differentiation, use of technology, reflection, and predictions which allows youth to rely on their strengths and take on appropriate challenges that nurture their development.

Career Exploration Connections

Lesson 8 asks students to brainstorm real-world examples of exponential functions and the Performance Task asks students to conduct research using different data sets that show growth or decay. Teachers are encouraged to expand these activities to include a career connections component to the research and brainstorming. Students can brainstorm what types of career fields or jobs need career fields or jobs need to have an understanding of growth and decay or the ability to graph exponential functions. Examples include: firefighters, accountants and bankers, scientists who study radioactive elements, biologists, engineers and builders, and ecologists. If students have difficulty brainstorming career fields, teachers can provide the careers and ask students to identify specific tasks for that job that require the use of exponential functions and an understanding of exponential growth and decay. Some examples include the following: Ecologists study growth and population, firefighters study burn patterns and predict the size and paths a fire could take, scientists perform radiocarbon dating to find the age of fossils. Teachers should encourage students to see that knowledge of exponential functions does play a role outside of the classroom and in many career fields.

For Technical Assistance with Empower Your Future connections and lessons, please request support by submitting a Coaching Request ticket using the Coaching Feature on TeachPoint.
POST-UNIT REFLECTION

On meeting the Learning and Language objectives

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Pre-Assessment on Exponent Rules
Lesson 1

DIRECTIONS: Complete the following activities.

1. Fill in the chart below:

<table>
<thead>
<tr>
<th>Exponent Form</th>
<th>Expanded Form</th>
<th>Simplification</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4x)^5</td>
<td>3 * 3 * 3 * 2 * 2 * y * y * y</td>
<td>36x^2</td>
</tr>
</tbody>
</table>

2. Determine if the equations below are true or false. If false, provide a correct solution and explain your answers.

   a. ________________________  (x^3)^4 = x^7

   b. ________________________ \( \frac{x^{15}}{x^7} = x^8 \)

   c. ________________________ (-3y^3)^2 = 3y^6

   d. ________________________ 2^3 * 2^2 = 2^5
3. Simplify and explain the differences between $x^3 \cdot x^3$ and $(x^3)^3$.

4. In the space below, draw a sketch of a box with a volume of $27x^3$ cubic centimeters and label its dimensions.

DIRECTIONS: Simplify the following expressions and circle the correct answers.

1. \( x^1 \cdot x^3 \)
   a) \( x^2 \)
   b) \( x^3 \)
   c) \( x^4 \)
   d) \( x^5 \)

2. \( x^1 \cdot x^4 \)
   a) \( x^2 \)
   b) \( x^3 \)
   c) \( x^4 \)
   d) \( x^5 \)

3. \( x^5 \cdot x^6 \)
   a) \( x^{11} \)
   b) \( x^1 \)
   c) \( x^2 \)
   d) \( x^{-1} \)

4. \( x^{-4} \cdot x^{-5} \)
   a) \( x^{-8} \)
   b) \( x^{-9} \)
   c) \( x^{-7} \)
   d) \( x^{-5} \)

5. \( x^{-8} \cdot x^8 \)
   a) \( x^1 \)
   b) \( x^0 \)
   c) \( x^2 \)
   d) \( x^8 \)

6. \( x^{-2} \cdot x^{10} \)
   a) \( x^{-12} \)
   b) \( x^{12} \)
   c) \( x^8 \)
   d) \( x^{-7} \)

7. \( x^{-7} \cdot x^3 \)
   a) \( x^2 \)
   b) \( x^3 \)
   c) \( x^5 \)
   d) \( x^4 \)

8. \( x^7 \cdot x^8 \)
   a) \( x^{14} \)
   b) \( x^{13} \)
   c) \( x^{15} \)
   d) \( x^1 \)
**Laws of Exponential Functions: Medium**

**Lesson 1—Extension**

**DIRECTIONS:** Simplify the following expressions and circle the correct answers.

<table>
<thead>
<tr>
<th>Expression</th>
<th>1. $\frac{x^2}{x^1}$</th>
<th>2. $\frac{x^4}{x^3}$</th>
<th>3. $\frac{x^5}{x^2}$</th>
<th>4. $\frac{x^6}{x^4}$</th>
<th>5. $\frac{x^5}{x^3}$</th>
<th>6. $\frac{x^7}{x^9}$</th>
<th>7. $\frac{x^4}{x^2}$</th>
<th>8. $\frac{x^5}{x^3}$</th>
<th>9. $\frac{x^3}{x^4}$</th>
<th>10. $\frac{x^7}{x^3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simplified Interpretation</td>
<td>$x^2$</td>
<td>$x^{2-3}$</td>
<td>$x^{5-2}$</td>
<td>$x^{6-4}$</td>
<td>$x^{5-3}$</td>
<td>$x^{7-9}$</td>
<td>$x^{4-2}$</td>
<td>$x^{5-3}$</td>
<td>$x^{3-4}$</td>
<td>$x^{7-3}$</td>
</tr>
<tr>
<td>Correct Answers</td>
<td>a) $x^2$</td>
<td>b) $x^{2-3}$</td>
<td>c) $x^{5-2}$</td>
<td>d) $x^{6-4}$</td>
<td>a) $x^2$</td>
<td>b) $x^{2-3}$</td>
<td>c) $x^{4-2}$</td>
<td>d) $x^{5-3}$</td>
<td>a) $x^2$</td>
<td>b) $x^{7-3}$</td>
</tr>
</tbody>
</table>
Pre-Assessment on Exponent Rules (pp. 4.10.1 to 4.10.2)

1. (p. 4.10.1)

<table>
<thead>
<tr>
<th>Exponent Form</th>
<th>Expanded Form</th>
<th>Simplification</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((4x)^5)</td>
<td>(4 \cdot 4 \cdot 4 \cdot 4 \cdot x \cdot x \cdot x \cdot x \cdot x)</td>
<td>(1024x^5)</td>
</tr>
<tr>
<td>((3)^3 \cdot (2)^2 \cdot y^3)</td>
<td>(3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot y \cdot y \cdot y)</td>
<td>(108y^3)</td>
</tr>
<tr>
<td>((6x)^2)</td>
<td>(6 \cdot 6 \cdot x \cdot x)</td>
<td>(36x^2)</td>
</tr>
</tbody>
</table>

2. (p. 4.10.1)

2a. F  
2b. T  
2c. F  
2d. T  

3. (p. 4.10.2)

\(x^3 \cdot x^3 = (x \cdot x \cdot x)(x \cdot x \cdot x) = x^6\)  
\((x^3)^3 = (x \cdot x \cdot x)(x \cdot x \cdot x)(x \cdot x \cdot x) = x^9\)

Extension—Laws of Exponential Functions: Easy (p. 4.10.3)

1. c  
2. d  
3. a  
4. b  
5. b  
6. c  
7. d  
8. c  

Extension—Laws of Exponential Functions: Medium (p. 4.10.4)

1. b  
2. b  
3. c  
4. c  
5. a  
6. c  
7. c  
8. a  
9. c  
10. d
Exponential Functions Organizer
Lesson 5

OVERVIEW: Exponential growth occurs when anything is increasing at a fixed percentage, such as 1% or 7%. Our global population has been growing exponentially. As a result, we have been consuming resources such as food, water, coal, oil and natural gas at exponential rates. But, we live on a finite planet and the exponential consumption of finite resources is not sustainable.

Take a look at the graphs below and answer these questions:
What do these graphs make you think about?
What are some potential consequences for our planet?
What questions do you have when you look at these graphs?

Adapted from: https://sustainabilityissues.wordpress.com/exponential-growth/
Why Exponential Functions Matter
Lesson 5

DIRECTIONS: Read the three scenarios completely and answer the questions provided.

Scenario One
Once upon a time, Zara lived next to Lake Washington in Seattle and had a beautiful garden. One day, Zara decided that she would save money on her water bill by watering her garden using water from the lake instead of her tap. Lake Washington is the 55th largest lake in the U.S., measuring about 22 miles long by 2 miles wide, so nobody would notice a small amount of water taken from it each day.

1. If Zara took one gallon of water from the lake each day, would that be sustainable? How long would it be before the lake was empty?
   
   *Let’s assume that there’s no rain or rivers feeding into the lake and nothing removing water except for Zara and that there are 770 billion gallons of water in Lake Washington.*

2. Can you create a graph and sample data table to represent your calculations?

Scenario Two
Zara sees a business opportunity here and begins to water her neighbors’ gardens, too. She keeps her expenses down by using water from the lake. She gains one new client every day and each client needs one gallon of water each day. Therefore, each day, she removes one additional gallon of water, so on the second day she uses 2 gallons per day and on the third day she uses 3 gallons per day, and so on.

3. Is this plan sustainable? How long will the water last?

4. Can you create a graph and sample data table to represent your calculations?

Scenario Three
Now Zara’s business really takes off. She gets one friend to be a client. Every day, each client gets one new friend to join. She goes from having 2 clients to 4 clients, then 8, 16, 32, 64, 128, etc. So, the amount of water she needs DOUBLES each day.

5. Is this plan sustainable? How long will the water last?

6. Can you create a graph and sample data table to represent your calculations?

Adapted from: https://sustainabilityissues.wordpress.com/exponential-growth/
Scenario One: Answer
Let's simplify the situation and ignore the hydrologic cycle (pretend that there's no rain or rivers feeding into the lake and nothing removing water except for Zara). Technically, she can't do this forever, because there is a finite amount of water in the lake. But, this is a big lake and one gallon is a small amount of water, so it's got to last a long time. Because there are 770 billion gallons of water in Lake Washington and she's removing one gallon each day, the lake will be empty in 770 billion days = 2.1 billion years. That's a long time (almost half the age of the earth).

Scenario Two: Answer
With Zara removing more water each day than the day before, it can't last as long as before. On day 1,240,967, Zara removes 1,240,967 gallons of water from the lake, but that is all that was left, so the water is gone. 1,240,967 days = 3,400 years. That's the era of ancient Egypt.

Scenario Three: Answer
Zara has now drained Lake Washington in a mere 40 days! That's the life span of a fruit fly. Such is the consequence of exponential growth.

Adapted from: https://sustainabilityissues.wordpress.com/exponential-growth/
Graphing Exponential Functions
Lesson 6

PART 1: Graph the functions for activities 1-4.

1. \( f(x) = 4^x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
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<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

2. \( f(x) = 0.5^x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td></td>
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<tr>
<td>-1</td>
<td></td>
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<td>0</td>
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<tr>
<td>1</td>
<td></td>
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<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>
3. \( f(x) = 1.25^x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
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<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

4. \( f(x) = 2^{\frac{x}{2}} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
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<tr>
<td>0</td>
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<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>
Graphing Exponential Functions
Lesson 6

PART 2: Without graphing, determine whether each function in the next four problems represents exponential growth or exponential decay. Explain your reasoning.

5. \( y = 129 (1.63)^x \)

6. \( f(x) = 2 (.065)^x \)

7. \( y = 12 (17/10)^x \)

8. \( y = 0.8 (1/8)^x \)
PART 3: Without graphing, determine whether each function in the final four problems represents exponential growth or exponential decay. Explain your reasoning.

1. \( y = 2 (1/5)^x \)

2. \( y = 24 (0.8)^x \)

3. \( y = 3 (6/5)^x \)

4. \( y = 7 (2/3)^x \)
Sustainability Project
Lessons 9-11

Name: ________________________________
Date: ________________________________

Project Packet
Exponential Functions Performance Task

Contents:
Project Introduction
   Overview
   Assessment Criteria
Process Steps
Sustainability Project
Lessons 9-11

PROJECT INTRODUCTION

According to the Environmental Protection Agency and environmentalscience.org, “the definition of ‘sustainability’ is the study of how natural systems function, remain diverse and produce everything it needs for the ecology to remain in balance. It also acknowledges that human civilization takes resources to sustain our modern way of life. To pursue sustainability is to create and maintain the conditions under which humans and nature can exist in productive harmony to support present and future generations.”

**SEE:** U.S. Environmental Protection Agency: Learn about Sustainability
www.epa.gov/sustainability/learn-about-sustainability#what

When studying environmental sustainability, certain themes and related issues are a part of a typical search. Some of those themes and issues are listed on this website:

**SEE:** Learning for a Sustainable Future
resources4rethinking.ca/en/resource-review-tool/issues

Use some of the ideas in the chart below to help you brainstorm a topic that interests you and that you would like to spend a little time researching:

<table>
<thead>
<tr>
<th>Themes</th>
<th>Issues</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air, Atmosphere, and Climate</td>
<td>Acid Rain, Air Pollution, Climate Change, Ozone Depletion</td>
</tr>
<tr>
<td>Ecosystems</td>
<td>Biodiversity, Carrying Capacity, Endangered Species, Habitat Loss, Interdependence, Invasive Species, Wildlife Protection</td>
</tr>
<tr>
<td>Food and Agriculture</td>
<td>Animal Rights, Aquaculture, Biotechnology, Conventional Farming, Food Security, Local Food, Organic Farming, Pesticides, Subsistence Farming</td>
</tr>
<tr>
<td>Human Health and the Environment</td>
<td>Access to Health Care, Environmental Contaminants and Health Hazards, Health Promotion, HIV/AIDS, Hunger and Malnutrition</td>
</tr>
<tr>
<td>Waste Management</td>
<td>Composting, Hazardous Wastes, Liquid Waste, Rethink-Reduce-Reuse-Recycle, Solid Waste Disposal, Source Reduction</td>
</tr>
</tbody>
</table>
Overview

Imagine that your class has received a request from the non-profit organization, Learning for a Sustainable Future (LSF). As a part of their classroom outreach, they are asking students to research a topic related to environmental sustainability and prepare a report that explains their findings and makes predictions or draws conclusions about environmental sustainability.

Each student should research a topic of his or her choosing, find an exponential data set, then create multiple representations of the data, analyze the data, and prepare a presentation that discusses the implications of the data with regard to sustainability. After the students receive peer feedback on their presentations, they will make adjustments before presenting to the entire class and passing in their final report.

Assessment Criteria

The assessment criteria may be used to develop a class rubric for assessing student presentations and final reports.

- Clear data set
- Identified independent and dependent variables
- Multiple representations of the relationship within the data (i.e., words, table, equation, and a graph)
- Explanation for all representations
- Showing of any of your math work in a neat and legible manner (e.g., calculations)
- Creation of a presentation as a video, PowerPoint (www.prezi.com is a nice free online alternative), speech, lesson, or demonstration. If you have other ideas, please ask!

- Introductory paragraph
- Concluding paragraph
Process Steps

Students may use the following process steps to help them organize their work as they are finding, analyzing, and interpreting their data.

1. Choose a topic and then research the topic and find an exponential data set using any of these resources:

   **SEE:** Statistic Brain
   www.statisticbrain.com
   Quantitative Environmental Learning Project
   seattlecentral.edu/qelp
   Socioeconomic Data and Applications Center
   sedac.ciesin.columbia.edu/search/data?contains=sustainability+data+sets

   The environmental sustainability topic I chose was: __________________________________________

   The website resource I chose was: __________________________________________

   Research and data:
   
   a. Explain what you have learned about your topic from your research:

   b. What data set have you chosen to analyze and interpret?

   c. What are the independent and dependent variables?
2. Organize your data and create multiple representations (words, table, equation, and graph) of the data.
   
   a. Explain what the data shows in words (using the vocabulary of exponential modeling):
   
   b. Create a table of the data.
   
   c. Create an equation that reflects the data set.
   
   d. Create a graph of the data (using the appropriate scales and units).
3. Write out the math calculations. Show every step clearly and neatly.
4. Write an introductory paragraph about the topic and the data you chose and why you chose it.

a. What environmental sustainability topic did you choose?

b. Why did you choose this topic?

c. What is the goal of this project?

d. What kinds of data did you discover?
5. Write a concluding paragraph about your topic and the data you chose.

   a. What does the data you chose tell us about sustainability?

   b. Are there any other factors that may have an impact on this relationship?

   c. How might these other factors influence the relationship?

   d. From the data, what predictions can be made about environmental sustainability?

   e. Based on your data, what recommendations would you make about further environmental action?

6. **Checkpoint:**
   Look back to see that you have included all of the assessment criteria. Is your work legible and neat?
Introduction

We are inundated with statistics all the time, and sometimes, we realize that these “facts” that we are being told often contradict each other. How can that be? How can the same data be represented in different ways and seem to show something different? How can people be manipulated into believing something based on data? In this unit on statistics, students will explore the most accurate ways to depict data and will see how statistics can be used to influence people by not showing the entire picture or by not using the most accurate measure of center.

The “Fundamentals of Statistics” unit is designed to come last in the Spring Season and will take about seven days to complete, with the option of making the unit a full two weeks by expanding the Performance Task and adding additional lessons on statistical bias and data transformations. These additional lessons will help students further their understanding of how data can be represented in ways that don’t show the most accurate depiction and will allow students to look at real-life examples of how data can be manipulated.

In order to engage students with the content of the unit, teachers should focus discussions around the Essential Questions:

- How can people use statistics to misrepresent data? Why is it important to collect and analyze data?
- To explore these questions, students will engage in activities where they collect and analyze data and will analyze statistics that seem to misrepresent data. They will be asked to decide how to most accurately depict data that they collect and to explain why they chose to represent their data in the way that they did.

The “Fundamentals of Statistics” unit primarily focuses on the standards:

- **S-ID.1**: Represent data with plots on the real number line (dot plots, histograms, and box plots).
- **S-ID.2**: Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.
- **S-ID.3**: Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).

To meet these standards throughout the unit, students will create and analyze numerous data samples. They will learn how to accurately represent data by creating dot...
plots, histograms, and box-and-whisker plots. They will also understand how to interpret skew in a set of data and will understand how outliers affect data. Students will walk away from this unit with a greater understanding of how data can be manipulated and how to represent data in an accurate and concise way.

Activities have been included in this guide, such as the Reaction Time Performance Task, in order to engage students with statistical analysis in a hands-on way. Whenever possible, the teacher should provide students with data sets that relate to their own lives and interests. Examples provided here, such as text message use and income, are meant to interest students, but the teacher can use different data sets if the students are interested in exploring other data.

In order to be successful in this unit of study, students must have the prerequisite skills of being able to find the mean, median, and mode of a set of numbers. If students need additional practice with these skills, the teacher will need to add in an additional day of practice. The teacher should also consider posting these definitions around the classroom to remind students of the meanings of these words.

Students might have difficulty conceptualizing the idea that the same set of data can be represented in different ways and can misrepresent the truth. Teachers can help students by showing them numerous examples of how this happens. In the first lesson, two extension readings have been provided that can be used to show students how statistics can be used to misrepresent information. The teacher may want to find more examples of this.

For adaptation ideas for this unit, see p. 4.11.3 on the right
### Statistics: Fundamentals of Statistics
Adapting This Short-Term Unit for Long-Term Programs

#### Plan 1 (Short)

| SPRING SEASON—Fundamentals of Statistics: Short-Term Programs |
|-------------|----------------|----------------|----------------|----------------|
| MONDAY | TUESDAY | WEDNESDAY | THURSDAY | FRIDAY |
| **Week 1** | | | | |
| Lesson 1: Introduction to Statistics | Lesson 2: Data Representation | Lesson 3: Data Representation with Box-and-Whisker Plots | Lesson 4: Skew and Shape of Data Distribution | Lesson 5: Standard Deviation |
| **Week 2** | | | | |
| Lesson 6: Reaction Time Project (Day 1) | Lesson 7: Reaction Time Project (Day 2) | | | |

#### Plan 2 (Long)

| SPRING SEASON—Fundamentals of Statistics: Long-Term Programs |
|-------------|----------------|----------------|----------------|----------------|
| MONDAY | TUESDAY | WEDNESDAY | THURSDAY | FRIDAY |
| **Week 1** | | | | |
| Lesson 1: Introduction to Statistics | Extension A: Samples and Bias | Lesson 2: Data Representation | Lesson 3: Data Representation with Box-and-Whisker Plots | Lesson 4: Skew and Shape of Data Distribution |
| **Week 2** | | | | |
| Lesson 5: Standard Deviation | Extension B: Data Transformations | Lesson 6: Reaction Time Project (Day 1) | Lesson 7: Reaction Time Project (Day 2) | Extension C: Reaction Time Project (Day 3) |

For a short-term program settings, the unit can stay as is (Plan 1).

For use in long-term settings the adaptation adds an additional day of introductory material (Extension A), which allows the teacher to discuss types of samples and bias in statistics. This will give students more time to look at real-life data and to discuss how knowing the type of sample helps people interpret data more accurately. Another new lesson (Extension B) on Data Transformations can be added after students learn about Standard Deviation. This will show students how a transformation of data using addition or multiplication will change the measure of center and the variation. Teaching this additional lesson offers teachers an opportunity to bring in real-world examples such as:

Would you rather have a 3% raise or a $2,000 increase in your salary?

The long-term adaptation also allows for an extra day (Extension C) for the Performance Task. Extending the Performance Task allows students to compare each other's data center and variability and lets students create and work with larger data sets.
UNIT PLAN

Fundamentals of Statistics
Designed by: J. Greenough and K. Miele
Theme or Content Area: Algebra 1—Statistics
Duration: 7 Lessons / 7 Days

Emphasized Standards (High School Level)

STATISTICS AND PROBABILITY

S-ID.1: Represent data with plots on the real number line (dot plots, histograms, and box plots).
S-ID.2: Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more data sets.
S-ID.3: Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).

Essential Questions (Open-ended questions that lead to deeper thinking and understanding)

How can people use statistics to misrepresent data?
Why is it important to collect and analyze data?

Transfer Goals (How will students apply their learning to other content and contexts?)

Students will use their knowledge of statistics to help them make an informed decision.
Students will apply their knowledge of statistics to compare and contrast data, synthesize data, and prepare presentations.

For Empower Your Future Connections, see p. 4.13.1
**Learning and Language Objectives**

By the end of the unit:

KUDs are essential components in planning units and lessons. They provide the standards-based targets for instruction and are linked to assessment.

<table>
<thead>
<tr>
<th>Students should know...</th>
<th>understand...</th>
<th>and be able to...</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mean</strong></td>
<td>Data can appear differently depending on which measure of center is used.</td>
<td>Decide when it is appropriate to use each measure of center.</td>
</tr>
<tr>
<td><strong>Median</strong></td>
<td>The most accurate measure of center will depend on the set of data being interpreted.</td>
<td>Compute mean, median, and mode.</td>
</tr>
<tr>
<td><strong>Mode</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Measures of center</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Standard deviation</strong></td>
<td>Standard deviation can be used to express the variability of the data and to compare two sets of data.</td>
<td>Interpret the distribution of data in order to determine which measures of center are most effective.</td>
</tr>
<tr>
<td><strong>Spread</strong></td>
<td></td>
<td>Find the range and the standard deviation of the data.</td>
</tr>
<tr>
<td><strong>Range</strong></td>
<td></td>
<td>Use standard deviation to compare two sets of data.</td>
</tr>
<tr>
<td><strong>Variability</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Histograms</strong></td>
<td>The most effective way to represent data will depend on the size and variability of the data.</td>
<td>Represent data on a number line.</td>
</tr>
<tr>
<td><strong>Dot plots</strong></td>
<td></td>
<td>Create and interpret data representations.</td>
</tr>
<tr>
<td><strong>Box-and-whisker plots</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Interquartile range</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Outliers</strong></td>
<td>Outliers can skew a set of data and cause misinterpretation.</td>
<td>Account for the effect of outliers.</td>
</tr>
<tr>
<td></td>
<td></td>
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</tr>
<tr>
<td><strong>Shape</strong></td>
<td>The shape of data is a tool to interpret data.</td>
<td>Evaluate the shape of their data.</td>
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<td><strong>Symmetric</strong></td>
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<tr>
<td><strong>Left- and right-skewed data</strong> (positive, negative)</td>
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</table>
Assessment Evidence
Quality questions raised and tasks designed to meet the needs of all learners

Performance Task and Summative Assessment (see pp. 4.12.19-4.12.21)

Aligning with Massachusetts standards

In the Performance Task, students will create data, represent it in various ways, compute the measures of center, and interpret the data’s shape and variance. They will use their knowledge of statistics to help them make an informed decision about how to present their data to their peers and teacher.

Using the “Reaction Timer” on the NRICH website, students will create two sets of data, one a baseline of their reaction time and one a measure of their reaction time with an option (distractor) that students have chosen. The final project will culminate in a presentation, during which students will present both sets of data and interpret them individually and in comparison to each other. See “Reaction Time Final Project” in the Supplement on p. 4.14.15 for detailed instructions.

SEE: Reaction Timer
https://nrich.maths.org/6044

Pre-Assessments (see pp. 4.12.7-4.12.12)

Discovering student prior knowledge and experience

Lesson 1: Do Now—What is measurable?
Students will review the types of data that are measurable and will determine what information they could collect from people that they cannot measure. (For example, we could collect everyone’s cell phone numbers, but we can’t measure that data. We could measure how many text messages each person sends per day.)

Lesson 2: Review of Mean, Median, Mode
Students will review how to find the mean, median, and mode for a set of data. The mean is the average, the median is the middle number when all data is arranged from low to high, and the mode is the number that appears most often.

Formative Assessments (see pp. 4.12.9-4.12.19)

Monitoring student progress through the unit

Lesson 2: Histogram, dot plot, and Exit Ticket
Lesson 3: Sports team box-and-whisker plot and Exit Ticket
Lesson 4: Box-and-whisker plots and Exit Ticket
Lesson 5: Exit Ticket

For Empower Your Future Connections, see p. 4.13.1
Multiple Means of Engagement

This is the *why* of learning. It is what makes students engage or disengage. Throughout the unit plan, the student will be provided with as many choices in the level of challenge and complexity as possible in order to recruit and sustain engagement. For example, the teacher will encourage and support students in setting their own personal, academic, and behavioral goals. The teacher will use many strategies to guide students, including reminders, guides, rubrics, checklists, and prompts among other things that focus students on self-regulatory goals. Student tasks will be varied, allowing for active participation, exploration, and experimentation. The teacher will provide differentiated models, scaffolds, and feedback, as well as content material that is culturally relevant and responsive to students’ backgrounds. Most important is that teachers design assignments/tasks with authentic outcomes that are purposeful and convey meaning to real audiences.

The lessons in this unit are designed so students are engaged in the mathematical concepts and “do” them along with the teacher. Students will work with data that is representative of real-world situations and will create data of their own to work with. Real-life discussions about statistics will take place throughout this unit of study to engage students in the “why” of learning. They will have a basic understanding of how statistics can be manipulated and will learn to question and evaluate statistics that they see. Students should conduct a Google search on some terms, such as histogram. This unit also may use resources, such as *Ted Talks*, to engage youth and make real-world connections. Evaluative emphasis should be placed on process, effort, and improvement. Formative Assessments are designed to invite personal response, self-evaluation, and reflection. Students should be given problems and be shown math examples that address a wide range of diversity and learning profiles in the classroom. Where possible, work should be done in pairs and/or small heterogeneous cooperative learning groups.

Modifications intended to adjust the unit’s learning and language objectives, Transfer Goals, level of performance and/or content will be necessary for students with mandated specially designed instruction described in their Individualized Education Programs (IEPs).

Multiple Means of Representation

This is the *what* of learning. There are many pathways to conveying information to students. Throughout the unit, the teacher will provide information and materials in several modalities such as diagrams, vocabulary cards or word sorts, word walls, posters, and charts with formulas for calculations; and models, videos, and audio for text. The teacher will also demonstrate concepts through hands-on activities such as the “Reaction Timer.”

The way information is displayed should vary, including size of text, images, graphs, tables, or other visual content. Information should be chunked into smaller elements, and complexity of questions can be adjusted based on prior knowledge competency. Reference sheets for examples, notes, vocabulary, and definitions can be differentiated for content.
Multiple Means of Action and Expression

This is the how of learning. In learning activities students will be provided options for demonstrating what they know and can do. Students will have options of writing or verbalizing their learning in Do Now activities or Exit Tickets. Students will have access to assistive technology if needed to support their learning. For example, students will have access to word processors with grammar checks, word prediction, and spell checkers. The teacher will also break down long-term goals into short-term reachable goals.

Performance Tasks can be differentiated by content, process, or product to address various learner profiles. Students will have some choice in the Performance Task regarding which distractor they want to use and how they will represent their data and present it to the class. Students will learn to create box and whisker plots on graph paper, but the teacher can also show students how to create these on graphing calculators. Opportunities for collaboration and whole-class discussion should be provided as needed.

Literacy and Numeracy
Across Content Areas

Reading
Students will read the instructions carefully in order to complete all assessments. Students will also read and interpret a variety of math texts, such as data sets and graphs, in order to solve problems or create solutions. Through extension activities, students have the opportunity to read about real-life applications of the material they are studying.

Writing
Students will engage in writing activities through “Do Now” activities or “Exit Tickets.” They will also need to write to express their reasoning and to prove their answers.

Speaking and Listening
Students will speak with their teacher and classmates in order to complete all of the assignments in this unit. Students should share their reasoning with their classmates and build on the ideas of their classmates to clarify their own thinking. For the Performance Task, students will create a presentation for their classmates that explains the data they collected, interprets the data, and explains why they represented the data the way that they did.

Language
Students will use accurate vocabulary to explain the mathematical concepts discussed in this unit. They will work with vocabulary by doing a word sort activity, and/or creating a word wall or word splash using the computer.

Numeracy
Students will work with data sets and graphic representations of data. They will create data sets and interpret data.
Resources (in order of appearance by type)

Websites

Lesson 1 (Extension)


Lesson 2 (Supplement to lesson)

Lesson 3
www.alcula.com/calculators/statistics/box-plot/

Lessons 5/6


Materials

Lesson 2: Data Representation—Dot Plot and Histogram Activity Sheet p. 4.14.1
Lesson 2: Data Set 1 Activity Sheet p. 4.14.2
Lesson 2: Data Set 2 Activity Sheet p. 4.14.3
Lesson 2: Data Representation Exit Ticket Activity Sheet p. 4.14.4
Lesson 3: Box-and-Whisker Plot Examples Activity Sheet p. 4.14.5
Lesson 3: Salesperson of the Year Activity Sheet p. 4.14.6
Lesson 3: Box-and-Whisker Plot Data Set Activity Sheet p. 4.14.7
Lesson 3: ANSWER KEY—Box-and-Whisker Plot Answer Key p. 4.14.8
Lesson 4: Mean Salary Activity Sheet p. 4.14.9
Lesson 4: Beech Street—Data Set 1 Activity Sheet p. 4.14.10
Lesson 4: Beech Street Data Set Practice Histogram Activity Sheet p. 4.14.11
Lesson 4: Beech Street—Data Set 2 Activity Sheet p. 4.14.12
Lesson 5: Standard Deviation Sample Data Sets 1 and 2 Activity Sheet p. 4.14.13
Lesson 5: Standard Deviation Steps Activity Sheet p. 4.14.14
Lesson 6: Reaction Time Final Project Activity Sheet p. 4.14.15
PREREQUISITES: Math skills needed for this unit

Students should be familiar with different types of graphs and charts (teachers will review these throughout the unit, too).

Students should know:

- How to calculate mean, median, and mode

Outline of Lessons

Introductory, Instructional, and Culminating tasks and activities to support achievement of learning objectives

INTRODUCTORY LESSON

Stimulate interest, assess prior knowledge, connect to new information

Note: Teachers should vet all videos included in this unit according to program standards and create templates or graphic organizers for students to monitor their comprehension of the material.

Lesson 1

Introduction to Statistics

Goal
Students will determine what kind of data is measurable.
Students will calculate mean, median, and mode.
Students will determine the range of data.

Do Now (time: 10 minutes)
The teacher will put a list of eight things on the board: phone numbers, hair color, number of text messages sent daily, income, number of products sold by a store, age, stress level, and happiness.
The teacher will ask students:

Which of these things are measurable? Which are not? Why?

Students will make lists on their own, then discuss their thoughts with partners. They will then discuss their answers with the class.

Note: There could be some disagreement among the students. For example, some students might think that you can measure stress levels or happiness on a scale. Allow the students to discuss this and how scales aren’t the best measure of data because people’s interpretation of happiness or stress may differ.
Hook (time: 5 minutes)
The teacher will show students a jar of marbles (or any other object). Students will guess how many marbles are in the jar. The teacher will record these estimates on the board.

Presentation (time: 20 minutes)
The teacher will use the data on the board to review with students how to find the mean, median, mode, and range of the data.

Mean = average
Mode = most common number
   (there might not be a mode in this data set, or there could be more than one)
Median = the middle number of the sorted data (if there is an even number of data items, the median is the average [mean] of the two middle values)
Range = maximum-minimum

The teacher will add these words to a word wall in the classroom and will instruct students to create a T-chart for new vocabulary words. They will write the word and definition on one side of the T-chart and will draw a representation of it on the other side.

The teacher will ask students:
   Are any of these numbers useful?
   What do these numbers tell us?
   Do you think that any of these measures of center are close to the actual number of marbles?
   Would you trust one of these measure of center more than you would trust your own guess?

Practice and Application (time: 15 minutes)
The teacher will tell the students how many marbles were actually in the jar. Students will calculate how far off they were and how far off the class was (from the mean, median, and the mode). Students will compare the distance from the mean, median, and mode and think about what that means. They will record their calculations on a piece of paper.

The teacher will then tell students to imagine that someone in the class guessed an outlier that was 100 over the actual number of marbles. With partners, students will find the mean, median, and mode of the original class data, plus our new number.

   How does this affect our mean?
   How is it different from the way that it affected the median?
   Which measure of center do you trust the most now?

Students will discuss their thoughts with partners. The teacher will ask students to think about how much an outlier affects the mean, median, and mode.

   Would it have a larger effect depending on how many pieces of data we were using?

For example, if we only had four pieces of data that we were using and one was an outlier, how much would that affect the mean? If we had 100 pieces of data and one was an outlier, how much would that affect the mean? The teacher will ask students to consider these questions for the mode and median as well.
Review and Assessment (time: 5 minutes)

Students will share their new answers with the class and their answers to the discussion questions. Students should note that the mean is affected most when there is an outlier in a small data set. For example, if we had a data set of 20, 22, 24, and 22, and an outlier of 102, our mean is going to be affected. Without the outlier, the mean is 22. With the outlier, the mean is 47.5. If we had 100 pieces of data that we were averaging with only one outlier, the mean would not be affected as much.

Note: If students had trouble finding the median, mode, and mean of the new data set, more time should be spent reviewing this.

Extension

The teacher could find a reading for the class about the difference of what U.S. income looks like based on whether we look at mean, median, or mode. An example that could be used as a starting point for class discussion is “Perception Vs. Reality On The U.S. Economy: The Median And The Mean”

SEE: Perception Vs. Reality On The U.S. Economy: The Median And The Mean
www.forbes.com/sites/bobmcteer/2014/11/05
/perception-versus-reality-the-median-versus-the-mean/#274f17623822

Note: This article is from 2014. After reading the article, students could research more up-to-date information about income in the United States. The teacher should create a notetaking guide for students, focusing on how the mean, median, and mode can be used to manipulate our view of statistics. The teacher should provide vocabulary guidance for words such as paradox, disproportionately, perception, as well as math terms for review: mean, median, mode. Another article, could also be used:

SEE: How to Lie with Statistics
www.psychologytoday.com/blog/the-fair-society/201111/how-lie-statistics

Again, teacher should provide a notetaking guide and vocabulary guidance.

INSTRUCTIONAL LESSONS

Build upon background knowledge, make meaning of content, incorporate ongoing Formative Assessments

Lesson 2

Data Representation

Goal

Students will create and interpret representations of data.

Do Now (time: 5 minutes)

The teacher will put the dot plot and histogram illustrations that follow on the board. The “Data Representation—Dot Plot and Histogram” Activity Sheet containing these two illustrations can be found on p. 4.14.1 of the Supplement.
The teacher will lead students through a think/pair/share exercise:

What do the Dot Plot and Histogram graphic representations of data tell us?

The teacher will then ask students:

What questions can you answer by looking at these graphic representations?

**Hook** (time: 5 minutes)

The teacher will tell students to survey 10 of their classmates to find out how many text messages each person sends and receives each day. (If 10 students are not available, ask students to imagine that they surveyed 10 people their own age to find out how many text messages they send or receive each day.)

Students will create a data table that represents the daily number of text messages that people send or receive.

**Presentation** (time: 15 minutes)

The teacher will create a sample set of data for people her own age so that she can model for the students what the dot plot would look like for people her own age. The teacher will use the information found in the illustration at left and adapted from the Pew Research Data to create the data set.

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**Dot Plot**

Fuel Economy for a Random Sample of 2015 Model Year Vehicles

![Dot Plot Image](adapted-from:www.mathbootcamps.com/how-to-read-a-dotplot/)

**Histogram**

Tree Heights

![Histogram Image](adapted-from:www.mathsisfun.com/data/histograms.html)

**Pew Research Data**

Daily Texts by U.S. Adults

<table>
<thead>
<tr>
<th>Age Group</th>
<th>Number of Daily Texts</th>
</tr>
</thead>
<tbody>
<tr>
<td>55+</td>
<td>2240</td>
</tr>
<tr>
<td>45-54</td>
<td>128</td>
</tr>
<tr>
<td>35-44</td>
<td>33</td>
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<tr>
<td>25-34</td>
<td>3853</td>
</tr>
<tr>
<td>18-24</td>
<td>1557</td>
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</tbody>
</table>

Monthly Texts by U.S. Adults

<table>
<thead>
<tr>
<th>Age Group</th>
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</tr>
</thead>
<tbody>
<tr>
<td>55+</td>
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<td>45-54</td>
<td>16</td>
</tr>
<tr>
<td>35-44</td>
<td>2240</td>
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<tr>
<td>25-34</td>
<td>33</td>
</tr>
<tr>
<td>18-24</td>
<td>75</td>
</tr>
</tbody>
</table>

Graphs adapted from an article at: www.textrequest.com/blog/many-texts-people-send-per-day/
The teacher will create a dot plot of her own data and show the students how to set it up and how to plot her data points. The teacher will ask students what they notice about her dot plot and what they wonder about it. Students should ask questions and interpret the data that is displayed in the teacher’s dot plot.

The teacher will then use a new data table with information for people aged 25-50 to create a histogram. The “Data Set 1” Activity Sheet on p. 4.14.2 in the Supplement provides a graphic organizer/table for this purpose. The teacher should give the table to students along with a blank histogram. While the teacher is explaining how to create a histogram with her data, the students will follow along and create their own, using the teacher’s data. The teacher will explain that when creating a histogram, the creator needs to decide on the intervals on the horizontal axis. The teacher will model how she decides on her intervals using a think-aloud.

**Practice and Application** (time: 20 minutes)

The students will create a dot plot based on the data set they created in the Hook. When creating the dot plot, students should title their dot plot and label the values. The teacher can give students circular stickers so that they quickly create their dot plots. Students should compare their dot plots with a partner and ask questions about the dot plots. If students were able to collect data from each other, they would want to think about the following questions:

- Do you think your data set is accurate? Why or why not?
- Did everyone in the class have a cell phone that they can use on a daily basis?

If students had to create their own data set, the teacher should ask them to consider questions such as:

- Why did you create the data that you did?
- Did you take into account people who don’t own cell phones? Why or why not?

Students will then be given a set of data based on actual cell phone usage for 18-24 year olds, and they will create a histogram with this new set of data. The “Data Set 2” Activity Sheet on p. 4.14.3 in the Supplement includes the data set for practice and application. Before creating the histogram, the teacher will ask students to use Google to find examples of bar graphs and histograms.

The teacher will ask students:

- What is the difference between these two types of graphs?

**Note:** The difference between a bar graph and a histogram is that a histogram groups numbers into ranges (or intervals), while a bar graph displays data at a singular value. The illustration at right on (p. 4.12.12) shows an example of the difference between a bar graph and a histogram.

The students will create their graphs on graph paper or on the “Data Set 2” Activity Sheet. The histogram should have a title, labels for both axes, and clearly numerated intervals. The teacher will add the words histogram and axes to the word wall, and students will add these words to their T-charts of new vocabulary terms, putting the word and definition on one side and a graphic representation on the other.

**Review and Assessment** (time: 10 minutes)

Students will post their histograms around the room and will walk around to look at the histograms that their peers created. Because the students’ interval ranges will be different and because the vertical scales will be different, the students’ histograms will look slightly different, but they should still be able to notice the general shape of the data distribution.
On an Exit Ticket, students should answer the following questions (see the “Data Representation Exit Ticket” on p. 4.14.4 in the Supplement):

- How does the histogram you created give you information not immediately obvious from the data table?
- What does the shape of the data distribution tell us about text messages sent by young people?

Compare your histogram to the one created from the teacher's data on older adults and text messages.
- How do they compare?

**Extension**

Students can use the data from their data set to find the **mean**, **median**, and **mode**. They can think about how the histogram gives us more information about the data set than simply the measures of center.

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**Lesson 3**

Data Representation with Box-and-Whisker Plots

**Goal**

Students will create and interpret box-and-whisker plots.
Students will determine the interquartile range from their plots.

**Do Now** (time: 5 minutes)

The teacher will project the box-and-whisker plot from the next page (illustration is also found in the “Box-and-Whisker Plot Examples” Activity Sheet on p. 4.14.5 in the Supplement). Before asking students to do anything with the plot, the teacher will add the term **box and whisker plot** to the word wall and will ask students to add it to their vocabulary T-charts.
LEARNING PLAN

Students will examine the box-and-whisker plot above and try to determine what the data is telling them. They will share their answers with partners before sharing their thoughts with the class. At the end of the discussion, the teacher will project the following altered box plot, which shows how the quiz results are distributed.

Hook (time: 5 minutes)
The teacher will give students a handout that shows (in a box-and-whisker plot) the monthly sales records of three employees who sell cars (Salesperson of the Year Activity Sheet on p. 4.14.6 in the Supplement). Students will have to decide who deserves the "Salesperson of the Year" award and be prepared to justify their decision. Students will share their answers with partners and try to convince their partners of their decision if partners disagree.

Presentation (time: 15 minutes)
The teacher will begin the presentation by projecting the Salesperson activity and asking students what the data shows and who deserves the award. Students likely will have different answers based on what they think is important.

Is the median the only thing that matters? Does a really bad sales day matter?
The teacher should allow the students to share their opinions as long as their interpretation of the data is correct. The teacher should correct any misinterpretations of data.

Next, the teacher will explain how to create a box-and-whisker plot using the data set in the "Box-and-Whisker Plot Data Set" Activity Sheet on p. 4.14.7 in the Supplement. When creating the plot, the teacher

Whisker plots adapted from www.softschools.com/math/topics/box_plots/
should use the content-specific vocabulary: median, lower quartile, upper quartile, interquartile range, box, whiskers, minimum, and maximum. The teacher should review how to find the median and how to find the quartiles. On graph paper or on the “Box-and-Whisker Plot Data Set” Activity Sheet, students should create the box-and-whisker plot with the teacher and label the parts as the teacher is explaining them. The teacher should also cover what to do if there were an outlier (such as a score of 32, for example). That should be plotted on the number line as a dot, not as part of the whisker. An ANSWER KEY is provided on p. 4.14.8.

**Practice and Application** (time: 25 minutes)
Students will research the ages of sports players on their favorite teams (3 different sports). If students do not have access to the internet, the teacher should print out rosters from a few different teams to allow students some choice in the teams they are comparing. Students will create three data tables (one for each team) and then turn the data tables into three box-and-whisker plots on the same grid. Students will label the number line. Students can use Excel to create their data tables or they can create these on paper. Students can also use the website *Acula Box Plot* to create the box-and-whisker plots. Students will have to copy and paste each box plot into a new document in order for them to all be on the same grid.

**Review and Assessment** (time: 5 minutes)
Students will post their box-and-whisker plots around the room. As an Exit Ticket, students will write down 3 things they notice about the age of people on different sports teams, 2 things the plots make them wonder, and 1 question they might want to research about the data that they or their classmates found.

**Lesson 4**

**Skew and Shape of Data Distribution**

**Goal**
Students will evaluate the shape of their data using various data representations. Students will recognize when data is skewed and will determine which measure of center and spread is most effective to use.

**Do Now** (time: 5 minutes)
Decide which job you want to take:

Job 1: Your interviewer tells you that in his company, the mean salary is $80,000. Only 8 people work here.

Job 2: Your interviewer offers you a job at the median salary, which is $30,000. Only 8 people work here as well.

Students will think about this, then talk with partners about their answers. The teacher will ask:

If you had the opportunity to ask your interviewer two questions to give you more information about your selection, what would they be?
**Hook** (time: 5 minutes)
The teacher will show students the “Mean Salary” Activity Sheet on p. 4.14.9 of the Supplement. With this new information, students will discuss what skewed the data and the decision that they made, since both scenarios refer to the same job.

**Presentation** (time: 10 minutes)
The teacher will show students the illustration below which shows two shapes of data distribution:

*Left-Skewed* (or negatively skewed) and *Right-Skewed* (or positively skewed)

While the teacher is reviewing the images, students should take notes in a T-chart. One side of the chart is for the type of data distribution with a verbal description, and the other side is for drawings of what that data would look like on a histogram and box-and-whisker plot.

![Graph showing left-skewed and right-skewed distributions](image)

**Practice and Application** (time: 30 minutes)
The teacher will explain to the students that they are being put in the role of “Statistical Advisor” for a neighborhood that recently had a subdivision added on to it. The teacher will explain to students that a subdivision is an area of land that has been divided out into plots of land to be sold individually for people to buy or for builders to build homes on. The residents of the neighborhood are curious how this subdivision is going to affect the average household income of their street’s neighborhood. To begin with, students will evaluate the average income of the neighborhood before the subdivision. Students will receive a set of data on household income in a neighborhood with 20 houses (Beech Street—Data Set 1 Activity Sheet on p. 4.14.10 in the Supplement). Some people are single and have only one income, some people are married and have two incomes. Some people may be unemployed or retired.

To evaluate this data, students will find the mean, median, and mode of this data set. The teacher will give students the “Beech Street Data Set Practice Histogram,” found on p. 4.14.11 in the Supplement, to fill
in with the data. The teacher will ask the students to notice the shape of the data; they should note that it is symmetric. They should also create a box-and-whisker plot of the data on the number line provided.

The teacher will then give the students a second data set that reflects the income of the new families that just moved into the new subdivision ("Beech Street—Data Set 2" Activity Sheet on p. 4.14.12 in the Supplement). The students should recalculate the mean, median, and mode.

How has this new data changed the data set as a whole?

Students should notice that the data is now right-skewed (or positively skewed). Now, the students need to represent the data.

What should they use?

Students should consider using a box-and-whisker plot because of the skewed nature of the data, and will make it on graph paper. The teacher will ask the students:

How does this box and whisker plot show skewed data differently than a histogram would?
Why would it be preferable?

Then, the students will imagine that a famous athlete, who makes $2 million dollars a year, just moved into this neighborhood. They will recalculate the mean, median, and mode.

Which measure of center changed the most from that one outlier?

The students should notice that the mean changed more significantly than the median. The teacher will ask the students which measure of center is therefore more appropriate for this set of data. The students should say the median. The teacher will show how to indicate the outlier on their box plot with a point rather than including it as part of the whiskers.

**Review and Assessment** (time: 5 minutes)

Students will look at the similarities between the box-and-whisker plots that they created from “Beech Street—Data Set 1” and then from the inclusion of “Beech Street—Data Set 2.”

What has stayed the same? What has changed? Students should use content-specific vocabulary when explaining what parts of the box-and-whisker plot have changed and what has stayed the same.

On an Exit Ticket, students will explain why a box-and-whisker plot is more appropriate than other representations to use when they have skewed data or extreme outliers.

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**Lesson 5**

**Standard Deviation**

**Goal**

Students will find the standard deviation of a data set and determine when to use it as an effective measure of variability.
**LEARNING PLAN**

**Do Now** (time: 5 minutes)
The teacher will post the following questions on the board:

Thinking about yesterday's class, what effect do skewed data and outliers have on your mean and median?

When you have outliers, which measure of center would you rather use—mean or median? Why?

Students should think about these questions on their own, then share their thoughts with partners.

**Hook** (time: 5 minutes)
Students will be introduced to the “Reaction Timer” on the NRICH website and test their reaction times (using their dominant hand) by clicking the star in the center of the screen to make it disappear, then clicking again when it reappears. They will do this twenty times, recording how many milliseconds it takes them each time to react to the star’s reappearance.

SEE: Reaction Timer
https://nrich.maths.org/6044

**Presentation** (time: 25 minutes)
The teacher will use her own data set for how long it took her to react to the reappearance of the star (or use the first data set in the “Standard Deviation Data Sets 1 and 2” Activity Sheet on p. 4.14.13 of the Supplement).

The teacher will show students her data set and show them the data in a histogram. The teacher will ask students to interpret the skew of the data. Students should note that the data is positively skewed (or right skewed). The teacher will ask students why this is the case. Students should mention that the teacher’s reaction time improved as she continued practicing the skill.

The teacher should then ask students which measure of center they should use. They should decide that the median is a better measure of center because the data seems to be skewed by the teacher’s learning curve of the activity.

The teacher will show students the median of her data set and she will ask students to create a box-and-whisker plot to express the variability. The teacher will ask students:

Why should we use a box-and-whisker plot to express this data?
What does it show?

The teacher will then show students a second set of data from the same activity (“Standard Deviation Data Set” Activity Sheet on p. 4.14.13 of the Supplement). She compiled this data after practicing the activity many times, so her data isn’t skewed. The teacher will ask students:

What measure of center should we use for this set of data? Why?

Students should say that they will use the mean because the data is mostly symmetrical.

The teacher will explain standard deviation to the class and will add this term to the word wall.

Students should also add it to their T-charts. The point of using standard deviation is to determine how far (on average) data points are from the mean. When comparing the standard deviation to the mean, we can see how much variability is in data.
Using the steps in the “Standard Deviation Steps” Activity Sheet on p. 4.14.14 of the Supplement, the teacher will lead the class through the process of finding the standard deviation of her second data set. Students should have a copy of the process of how to find standard deviation. After showing students how to find it, she will show them the formula below:

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2}$$

The teacher will explain what each symbol means and how it relates to the steps she took to find the standard deviation. As the teacher is reviewing what each symbol is, students will write the symbols on their T-charts with the definitions. The teacher will also add these symbols to the word wall near where she posted the standard deviation term so that students can refer to the word wall at any time.

For guidance on what each symbol means and for helping students understand the formula, the teacher should review the website below. The teacher might decide to allow students additional time to look over the website since it guides them through the process of finding standard deviation and explains what each symbol means.

**SEE:** Standard Deviation Formulas
www.mathsisfun.com/data/standard-deviation-formulas.html

The teacher will ask students:

How consistent was her reaction time in the activity?

In order to find this, the students need to compare the standard deviation to the mean and determine how much variability there is.

**Practice and Application** (time: 15 minutes)

Students will go back to the “Reaction Timer” on the NRICH website and practice the skill. When they feel confident, they should begin recording their reaction times (using their dominant hand) and collect 20 pieces of data in a row. They cannot discount a piece of data once they start recording.

**SEE:** Reaction Timer
https://nrich.maths.org/6044

Students should then practice with their non-dominant hand. Again, when they feel confident, they should begin recording 20 pieces of data in a row. They cannot discount a piece of data once they start recording.

Students will find the mean of each set of data and follow the steps of finding the standard deviation for each set of data.
Review and Assessment (time: 5 minutes)

Students will compare the mean and the standard deviations that they found in their left- and right-hand data. On an Exit Ticket, they will explain what the deviation and the differences in mean represent and why. They will compare the variability of their data with a classmate and determine whose data was most consistent with the mean. They will discuss why this might be. The teacher will review the Exit Tickets to check for understanding and will revisit this the next day if students did not understand standard deviation.

CULMINATING LESSONS

Includes the Performance Task, i.e., Summative Assessment—measuring the achievement of learning objectives

Lesson 6

Reaction Time Project (Day 1)

Goal

In preparation for the “Reaction Time Final Project,” students will create data sets, represent the data with histograms and box and whisker plots, and interpret the shape and variability of the data distribution.

Do Now (time: 5 minutes)

Students will review all terminology with a word sort. The teacher will use index cards to create the word sort.

One term will be on each index card. Terms should include:
- Dot plot, histogram, box-and-whisker plot
- Right-skewed, left-skewed, symmetric
- Mean, median, mode
- Range, standard deviation, interquartile range

Students will group the cards into these four categories:

- Types of Data Representation
- Distribution of Data
- Measures of Center
- Variability

Hook (time: 5 minutes)

The teacher will introduce the students to the options on the “Reaction Timer” on the NRICH website and will allow students to test out their reaction times using some of them.

SEE: Reaction Timer
https://nrich.maths.org/6044
Presentation (time: 10 minutes)
The teacher will pass out the final project to the class and will explain the expectations of the final project Performance Task. The “Reaction Time Final Project” Activity Sheet can be found p. 4.14.15 in the Supplement. The teacher will create a rubric with the class and students should be given a chance to self-assess or peer assess after their presentations. Students should say that a perfect final project should include: accurate computation, clear representation of data, appropriate interpretation of data, and an effective measure of variability.

Practice and Application (time: 25 minutes)
Students will begin work on the project. They will create their data sets and begin work on the various representations and calculations. The teacher should circulate around the room to answer any questions that students have. Students will have the option to use Excel to keep track of their data sets or to create a table using a pencil and paper. Their histograms can be created in Excel and the box and whisker plots can be created on the website “Acula Box Plot.” Students can do a Google search to find out how to use Excel to create histograms and follow the instructions that they find.

   SEE: Acula Box Plot
   www.alcula.com/calculators/statistics/box-plot/

Review and Assessment (time: 5 minutes)
As an Exit Ticket, students should write two things that they are doing well and one thing that they need help with. The teacher will review these Exit Tickets before the next class so that she can help students with whatever they are struggling with.

Lesson 7

Reaction Time Project (Day 2)

Goal
Students will create data sets, represent the data with histograms and box-and-whisker plots, and interpret the shape and variability of the data distribution.

Do Now (time: 5 minutes)
The teacher will check in with students based on their Exit Tickets from the day before and group them according to what they need help with.

Hook (time: 5 minutes)
The teacher will poll the class about which options (distractors) they used. The teacher will make a list on the board. Students will guess which distractor will cause the most variability and have the greatest effect on the mean and discuss why they think this is the case.

Distractor options are listed on NRICH website.

   SEE: Reaction Timer
   https://nrich.maths.org/6044
Presentation (time: 5 minutes)
The teacher will review the two ways to express the center and variability of a data set (with a box-and-whisker plot using interquartile range as variability OR the using the mean with standard deviation). The teacher will ask students:

When would you use each? Why?

Note: Students will create both with their data; however, they are going to present only one to the class as the best representation, so they need to understand which one to use for their data and why.

Practice and Application (time: 30 minutes)
Students will continue to work on their project and will create both representations. Although they will turn both in, they will need to decide which representation to display in their presentation to the class. Students will have the option to complete all work on paper or on a computer.

When students think they are done with the project, they should review their work against the rubric to make sure that they have included everything that they need to include.

Review and Assessment (time: 15 minutes)
Students will present their projects to the class, explaining what effect their distractor had on their reaction time, why they chose the representation that they did, what the shape of the data distribution is, and how much variability there is.
POST–UNIT REFLECTION

On meeting the Learning and Language objectives

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Connections to Empower Your Future

UNIT: Fundamentals of Statistics

Future Ready Connections

Along with the rubric designed for the Performance Task, teachers are encouraged to use the Future Ready Rubric to evaluate students’ growth and are encouraged to have students self-evaluate their progress using both rubrics. Youth have many opportunities to strengthen their communication and listening skills through partner work where they discuss and persuade each other, group discussions of the benefits of different methods for data representation, and the presentation of their Performance Task. Students will also communicate through writing in their Do Now and Exit Tickets which can be evaluated for clarity, coherence, and critical thinking. Youth should also be evaluated for initiative and self-direction as they complete Reaction Timer activities, the charting and graphing activities for their formative assessments, and their summative assessment.

Teachers should reflect on whether or not youth stay on task without prompting and if they push themselves to thoroughly complete each activity, answer their own questions, and create a detailed final product instead of only addressing the minimum required information. Teachers should encourage students to reflect on how they demonstrated growth and increased understanding throughout the unit, what they could do to further improve their skills and understanding, and how their learning is transferable to other situations and experiences.

Essential Question Connections

The two Essential Questions for this unit ask “How can people use statistics to misrepresent data?” and “Why is it important to collect and analyze data?” These questions encourage youth to understand the power of data to inform, persuade, sway, and shape conclusions and understandings of a situation or topic. Throughout the unit, students are encouraged to reflect on how small additions or exclusions of data points and the use of different methods for representing data can impact the interpretation of the data. Teachers can encourage youth to use their understanding of data and possible data manipulations to be savvy consumers and readers of data by asking them to review data reports with a critical eye. Students can conduct additional research on topics such as: cell phone networks with best service coverage, benefits of specific vitamins and supplements, cars with the best safety rating, among others.

Understanding how to critically assess data reports will allow youth to be savvy consumers of important goods and will allow them to make decisions about their lives based on objective data.

PYD/CRP Connections

This unit reflects Culturally Responsive Practice by using realistic situations and data sets for the histograms and box and whisker plots. Students see that math is a universal skill that can be used in the academic, professional, and personal realms and that the ability to collect, manipulate, and report data impacts many facets of their lives. Students can apply their knowledge about data collection, analysis, and reporting to many actions in their lives such as planning a major purchase, deciding on a candidate during an election, or even playing fantasy football.
The responsibility is placed on each youth to demonstrate their own learning and support the learning of their classmates which allows for interpersonal skill development.”

The application of math skills to meaningful and purposeful contexts allows for authentic outcomes that youth can practice and duplicate independently in the future.

The unit also demonstrates Positive Youth Development by encouraging students to be leaders in the classroom and to be active participants in their learning. Students must conduct experiments and chart results during the Reaction Timer activities in which they get to decide how to represent the data most effectively. Youth also have the opportunity to work with, support, and learn from each other’s discussions, presentations, and representations of data. The responsibility is placed on each youth to demonstrate their own learning and support the learning of their classmates which allows for interpersonal skill development.

Career Exploration Connections

Teachers can expand on the unit by making connections between the content and career exploration research. Students can research which career fields and organizations collect, analyze, and use statistics. Examples include: educational testing and measurements, Food and Drug Administration, finance, risk assessment, marketing, insurance, manufacturing, ecology, and many more.

Students can use search engines and keywords or visit the American Statistician Association (ASA) website to identify the career fields.

SEE: American Statistician Association
www.amstat.org/careers/
whichindustriesemploystatisticians.cfm

The ASA site provides links to different career fields and explains how that career field uses data. Teachers can have students conduct research, create a product (such as a PowerPoint presentation, brochure, poster, etc.) that summarizes the information, and present the information to the class. Students can also use the Massachusetts Career Information System (MassCIS) website to research jobs, career fields, industries, and programs of study associated with statistics and data analysis.

SEE: MassCIS
https://portal.masscis.intocareers.org

For Technical Assistance with Empower Your Future connections and lessons, please request support by submitting a Coaching Request ticket using the Coaching Feature on TeachPoint.
Data Representation—Dot Plot and Histogram

Lesson 2

DIRECTIONS: Students will study these examples of a Dot Plot and Histogram and use them for reference.

Dot Plot:

Fuel Economy for a Random Sample of 2015 Model Year Vehicles

Graph adapted from: www.mathbootcamps.com/how-to-read-a-dotplot/

Histogram:

Tree Heights

Graph adapted from: www.mathsisfun.com/data/histograms.html
Data Set 1
Lesson 2

DIRECTIONS: Students will create a histogram from the data provided in the table below.

| Daily Text Messages Sent and Received (Ages 25-50) |
|-------------|-------------|-------------|-------------|-------------|-------------|
| 33          | 70          | 56          | 47          | 32          | 62          |
| 16          | 79          | 72          | 54          | 56          | 62          |
| 81          | 46          | 55          | 28          | 78          | 10          |
| 70          | 56          | 2           | 78          | 51          | 67          |
| 13          | 32          | 87          | 0           | 90          | 49          |

Histogram
Data Set 2
Lesson 2

DIRECTIONS: Students will create a histogram from the data provided in the table below.

| Daily Text Messages Sent and Received (Ages 18-24) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 73              | 0               | 251             | 25              | 144             | 166             |
| 48              | 34              | 224             | 168             | 149             | 132             |
| 189             | 254             | 47              | 100             | 80              | 191             |
| 15              | 71              | 209             | 146             | 19              | 207             |
| 75              | 143             | 29              | 177             | 80              | 111             |

Histogram:
Data Representation Exit Ticket
Lesson 2

**DIRECTIONS:** Students will answer the questions below based on the histograms they created.

1. How does the histogram you created give you information not immediately obvious from the data table?

2. What does the shape of the data distribution tell us about text messages sent by young people?

3. Compare your histogram to the one created from the teacher’s data on older adults and text messages. How do they compare?
Box-and-Whisker Plot Examples

Lesson 3

DIRECTIONS: Students will study the box-and-whisker plot samples below.

Mrs. Sanchez Period 1 Math Results

Whisker plots adated from: www.softschools.com/math/topics/box_plots/
Salesperson of the Year
Lesson 3

DIRECTIONS: Below are the monthly sales records for the salespeople at Ken’s Auto Sales. Ken is trying to decide who deserves “Salesperson of the Year.” Who would you choose? Why?

Monthly Car Sales 2016

Giselle

Javier

Patty
**Box-and-Whisker Plot Data Set**

**Lesson 3**

**DIRECTIONS:** Students will create a box-and-whisker plot using the data set below.

<table>
<thead>
<tr>
<th>Ms. Pierce’s biology class test scores:</th>
</tr>
</thead>
<tbody>
<tr>
<td>83  68  69  94  85  78</td>
</tr>
<tr>
<td>83  71  73  97  80  88</td>
</tr>
<tr>
<td>83  77  70  75</td>
</tr>
</tbody>
</table>

**Box-and-Whisker Plot**
Sorted Data: (lowest to highest)
68, 69, 70, 71, 73, 75, 77, 78, 80, 83, 83, 85, 88, 94, 97

Median: 79
Lower Quartile: 72
Upper Quartile: 84
Minimum: 68
Maximum: 97
Interquartile Range: 72-84
Mean Salary
Lesson 4

You:
What would my starting salary be?

CEO:
I'll put it this way—our average starting salary is $80,000.

You: $30,000

All your coworkers

$30,000 $30,000 $30,000 $30,000 $30,000 $30,000 $30,000 $30,000

Average: $80,000

CEO’s son: $430,000
### Beech Street—Data Set 1

**Lesson 4**

<table>
<thead>
<tr>
<th>House Number</th>
<th>Yearly Household Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$41,000</td>
</tr>
<tr>
<td>5</td>
<td>$22,000</td>
</tr>
<tr>
<td>7</td>
<td>$48,000</td>
</tr>
<tr>
<td>8</td>
<td>$52,000</td>
</tr>
<tr>
<td>11</td>
<td>$86,000</td>
</tr>
<tr>
<td>13</td>
<td>$61,000</td>
</tr>
<tr>
<td>14</td>
<td>$54,000</td>
</tr>
<tr>
<td>18</td>
<td>$37,000</td>
</tr>
<tr>
<td>19</td>
<td>$55,000</td>
</tr>
<tr>
<td>20</td>
<td>$72,000</td>
</tr>
<tr>
<td>21</td>
<td>$78,000</td>
</tr>
<tr>
<td>24</td>
<td>$66,000</td>
</tr>
<tr>
<td>27</td>
<td>$69,000</td>
</tr>
<tr>
<td>28</td>
<td>$52,000</td>
</tr>
<tr>
<td>30</td>
<td>$59,000</td>
</tr>
<tr>
<td>32</td>
<td>$31,000</td>
</tr>
<tr>
<td>33</td>
<td>$57,000</td>
</tr>
<tr>
<td>34</td>
<td>$38,000</td>
</tr>
<tr>
<td>36</td>
<td>$45,000</td>
</tr>
<tr>
<td>38</td>
<td>$76,000</td>
</tr>
</tbody>
</table>

**DIRECTIONS:**

You’ve been asked to act as Statistical Advisor for the residents of Beech Street. Their small neighborhood is changing! A subdivision has been built on their street, and new neighbors are moving in. The residents want to know how this subdivision has affected their average household income. The data at left is the income of the residents before the subdivision was built.
Beech Street Data Set Practice Histogram
Lesson 4

DIRECTIONS: Use the blank histogram below to show the Beech Street data before the subdivision. Underneath the histogram, create a box-and-whisker plot of the data on the number line.
**Beech Street—Data Set 2**

**Lesson 4**

**DIRECTIONS:** The new data on the Beech Street subdivision is finally in, and it is your job as statistical advisor to represent and interpret it. Here is the information about the household income.

<table>
<thead>
<tr>
<th>House Number</th>
<th>Yearly Household Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>$120,000</td>
</tr>
<tr>
<td>40</td>
<td>$145,000</td>
</tr>
<tr>
<td>41</td>
<td>$95,000</td>
</tr>
<tr>
<td>42</td>
<td>$111,000</td>
</tr>
<tr>
<td>43</td>
<td>$98,000</td>
</tr>
<tr>
<td>44</td>
<td>$134,000</td>
</tr>
<tr>
<td>45</td>
<td>$108,000</td>
</tr>
<tr>
<td>46</td>
<td>$103,000</td>
</tr>
<tr>
<td>47</td>
<td>$88,000</td>
</tr>
<tr>
<td>48</td>
<td>$146,000</td>
</tr>
<tr>
<td>49</td>
<td>$158,000</td>
</tr>
<tr>
<td>50</td>
<td>$75,000</td>
</tr>
</tbody>
</table>
**Standard Deviation Sample Data Sets 1 and 2**

**Lesson 5**

**DIRECTIONS:** Study the data and histogram below. Interpret the skew of the data. What measure of center should be used?

Here is sample data for someone trying the Reaction Timer for the first time:

<table>
<thead>
<tr>
<th>Sample Data Set 1</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>497</td>
<td>481</td>
<td>477</td>
<td>372</td>
<td>403</td>
</tr>
<tr>
<td>418</td>
<td>370</td>
<td>358</td>
<td>372</td>
<td>345</td>
</tr>
<tr>
<td>372</td>
<td>440</td>
<td>388</td>
<td>359</td>
<td>391</td>
</tr>
<tr>
<td>340</td>
<td>337</td>
<td>373</td>
<td>396</td>
<td>386</td>
</tr>
</tbody>
</table>

**Histogram:**

Here is sample data for someone trying the Reaction Timer after some practice:

<table>
<thead>
<tr>
<th>Sample Data Set 2</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>348</td>
<td>341</td>
<td>407</td>
<td>267</td>
<td>330</td>
</tr>
<tr>
<td>390</td>
<td>398</td>
<td>317</td>
<td>292</td>
<td>439</td>
</tr>
<tr>
<td>346</td>
<td>350</td>
<td>357</td>
<td>423</td>
<td>396</td>
</tr>
<tr>
<td>334</td>
<td>393</td>
<td>321</td>
<td>404</td>
<td>375</td>
</tr>
</tbody>
</table>
### Standard Deviation Steps

**Lesson 5**

**DIRECTIONS:** Use these steps to find standard deviation.

<table>
<thead>
<tr>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard deviation is the square root of the variance.</td>
</tr>
</tbody>
</table>

**Note:** These are the steps for finding the standard deviation of an entire population.<br> If you need to find the standard deviation for a sample of a population, divide by \( n-1 \) instead of by \( n \). Since standard deviation is the square root of the variance, first you will find the variance.

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Step 1:</strong></td>
<td>Find the mean of your data.</td>
</tr>
<tr>
<td></td>
<td>( \mu )</td>
</tr>
<tr>
<td><strong>Step 2:</strong></td>
<td>Subtract the mean from each data value and square each of these differences. (These are the squared differences.)</td>
</tr>
<tr>
<td></td>
<td>( (x_i - \mu)^2 )</td>
</tr>
<tr>
<td><strong>Step 3:</strong></td>
<td>Find the mean of the squared differences by finding their sum and dividing by the count of data values. This is the variance.</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2 )</td>
</tr>
<tr>
<td><strong>Step 4:</strong></td>
<td>Take the square root of the variance. This is the population’s standard deviation.</td>
</tr>
<tr>
<td></td>
<td>( \sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2} )</td>
</tr>
</tbody>
</table>
Reaction Time Final Project
Lessons 6-7

In this final project, you will create data, represent it in various ways, compute the measures of center, and interpret the data’s shape and variance. You will use your knowledge of statistics to help you make an informed decision about how to present your data to your peers and teacher.

You will create two sets of data, one a baseline of your reaction time and one a measure of your reaction time with an option (distractor) that you have chosen. Your final project will culminate in a presentation, during which you will present both sets of data and interpret them individually and in comparison to each other.

You should follow the four steps for both your standard reaction time (without any options) and your reaction time with an option selected.

Step 1: Create your data set. You should have 40 data values for your reaction time. Remember to take 40 consecutive values; once you begin your data recording, you may not discount any value.

Step 2: Find the mean, median, mode, and range of your data.

Step 3: Represent your data with both a histogram and a box and whisker plot.

Step 4: Interpret the shape and variance of your data and be prepared to discuss it in your presentation. Consider which representation you will use to show the data of your reaction time. Will you choose a histogram or a box and whisker plot? Why? Consider how you will describe the center and variability of your data. Will you use the mean and standard deviation, or the median and the interquartile range? Why?

After all data has been represented and interpreted, consider the two data sets together. How did the option you chose affect your reaction time? How will you use your data representation or interpretation as evidence of that effect?

For your final project to be considered “perfect,” you should have:

- accurate computation
- clear representation of data
- appropriate interpretation of data
- an effective measure of variability

You will create a rubric for these categories with your class and teacher.

You may create your data sets by drawing your data tables on paper or by using an Excel spreadsheet. Likewise, you may create your histograms and box and whisker plots on the computer in Excel and by using “Acula Box Plot.” You may use Google to find out how to create your histogram in Excel.

SEE: Acula Box Plot
www.alcula.com/calculators/statistics/box-plot/
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Introduction to Geometry

Geometry is more than memorizing formulas or finding surface area. Geometry is about equipping students with the critical thinking and problem-solving skills needed to succeed in the 21st Century.

Introduction

When students think about studying geometry, they most likely think about memorizing formulas and writing proofs—skills that they don’t believe they will ever use once they leave high school.

When will they ever need to find the volume of a sphere or the surface area of a box?

When will they ever need to write a proof for a mathematical theorem?

These are the types of questions that geometry teachers are plagued with whenever a new topic of study arises, but teachers and students should rest assured that the skills that are being taught in geometry are skills that students will need—and use—throughout their lives. As University of Cambridge lecturer Piers Bursill-Hall states, “Studying geometry reveals—in some way—the deepest true essence of the physical world. And teaching geometry trains the mind in clear and rigorous thinking” (Bursill-Hall).

Throughout the study of geometry, students will engage in basic mathematical skills of angle measurement, learn how to construct a mathematical argument, and explore how formulas are derived. Most importantly, perhaps, geometry teaches students how to reason. When students begin writing proofs, they must reason their way through the writing of them. It is not enough to say that a theorem is true simply because the textbook says it is; students must learn to articulate why that theorem is true and follow a logical series of steps to prove that it is true. What teachers of geometry are teaching, then, is truly how to think.

One of the ways that students can be engaged in the study of geometry is to explore the many career paths that they might pursue that involve geometry. Some career paths that utilize the skills and knowledge of geometry are:

- Jewelers who must cut specific shapes into gems
- Fashion designers who cut patterns from fabric and want to make the most clothing from the least material
- Construction workers who must understand how to stabilize their structures
- Optical lens makers who use geometry to ensure cameras will capture images
- Robotics engineers who design robots and assembly lines
- Plumbers who need to use angle measurements to fit pipes correctly
- Astronomers who study the patterns of stars and planets
- 3D graphic artists who create their art with geometric shapes (“Jobs”)

“What teachers of geometry are teaching, then, is truly how to think.”
“Pointing out these real-life connections to students will help them see why learning geometry is so essential.”

The list of career paths that involve the skills learned in geometry could go on and on, but since students might not be entering their careers for a few years, they might be more interested in exploring the ways that they encounter topics studied in geometry in their daily lives right now. From the creation of packaging for foods that they eat, to the angles in buildings, to parallel lines painted in parking lots, geometry is all around us. Pointing out these real-life connections to students will help them see why learning geometry is so essential.

The skills and concepts that are the basis of geometry have been around for thousands of years. The same formulas that were used to build the ancient pyramids of Egypt are being used today by video game makers to create the games that our students are playing. Students might be interested to know that the popular computer game “Minecraft” is completely based on the study of geometry since it uses geometric shapes to create virtual worlds (“Jobs”).

It is unclear how many new applications for these formulas and skills await in the future, but if we equip our students with geometric skills, they will be able to pave the way for the future of applying geometric concepts to life in the 21st century.

Geometry Course Content

One of the key shifts in the Common Core State Standards (CCSS) for Mathematics asks teachers to increase rigor in their classrooms. This rigor does not refer to making math more challenging for students or to introducing high-level skills at an earlier age. Instead, it asks teachers to implement what the CCSS calls three elements of rigor: conceptual understanding, procedural skills and fluency, and application.

While all three elements of rigor are addressed in the following units, teachers should pay careful attention to the “application” element of the CCSS, which asks “students to use math in situations that require mathematical knowledge. Correctly applying mathematical knowledge depends on students having a solid conceptual understanding and procedural fluency” (“Key Shifts”).

As much as possible, teachers will ask students to apply their new understandings to new situations and real-life tasks. The Summative Assessments of each unit should ask students to engage in rigorous tasks where students will apply their newfound knowledge to a new task that replicates something that they might encounter in their daily lives. As the CCSS states, “Students must be able to access concepts from a number of perspectives in order to see math as more than a set of mnemonics or discrete procedures” (“Key Shifts”).

By applying their learning to new scenarios, students will ensure mastery of the knowledge and skills addressed in the CCSS.

The Geometry course addresses several Essential Questions that students should think about throughout the year:

- How do we develop mathematical conjectures?
- How do we use reasoning and logic to solve geometric proofs?
- How do we construct a convincing argument?
- How does the Pythagorean theorem help us solve real-world problems?
- How are various formulas derived?

While many of these questions are best suited to be discussed within a certain season, most can be discussed throughout the entire year and in other disciplines. Just as teachers of English teach students to create convincing arguments using reasoning and logic, teachers of math will instruct students to create convincing arguments using mathematical data to support their claims. Embedded within all three units are the Common
Core State Standards for Mathematical Practice. These standards include asking students to reason abstractly and quantitatively, to construct arguments and critique the reasoning of their classmates, and to use appropriate tools.

Teaching Geometry in DYS Schools

Whether students are just beginning their study of Geometry in the “Essentials of Geometry” or finishing the year with their study of “Measurement,” teachers need to ensure that students can connect what they are learning in the classroom to their everyday lives. Whenever possible, teachers should provide students with real-life examples of when they will use the mathematical skills that they are learning in the classroom so that students see a purpose in what they are learning. Teachers can provide these examples through readings, job connections, or through Summative Assessments that ask students to solve real-world problems. Throughout the sample units, examples have been provided for teachers to encourage this discussion.

Using a variety of instructional methods will encourage student participation. Teachers will provide students with as many hands-on activities as possible so that students can use manipulatives to engage in the skills they are learning. Teachers will also provide students with videos, games, and readings to encourage learning through a variety of media. Additionally, teachers will want to use technology as much as possible to give students the skills that they will need to be successful.
in future careers. Many websites have been provided for teachers to use to engage students in the content and skills of this course.

The study of Geometry can be challenging for many students because of the number of equations and vocabulary words that students are encountering for the first time. Teachers will create word walls and will have students create personalized reference sheets to aid them while they acquire this knowledge. While it is important that students learn many of these equations and new vocabulary terms, the emphasis should not be on simply memorizing the words and equations, but should be on using the equations to solve problems and using the newly acquired vocabulary to articulate what they are learning and how they are solving problems.

Works Cited


Reading the Geometry Scope and Sequence Chart

The amount of information contained in the Scope and Sequence on the following pages may seem overwhelming at first. The best way to study it is to read across from left to right. The keys below on this page offer guidance on how to properly access the Scope and Sequence Chart on pp. 5.2.2 to 5.2.3.

The Scope and Sequence is COLOR-CODED. Each color is important, and its meaning and the main highlights in DYS pedagogy are described in the key in the LEFT column below. The RIGHT column shows the Geometry topics and the seasons when they are taught during the academic year.

Topics listed with an asterisk (*) in the Scope and Sequence have exemplar units in this Guide.

### Scope and Sequence Chart Key

- **The GOLD header rows identify columns for Topics, Emphasized Standards, Essential Questions, Transfer Goals, and Performance Assessments for each of the seasons in mathematics.**

- **The GREEN row contains the FALL season.**

- **The BLUE row contains the WINTER season.**

- **The RED row contains the SPRING season.**

- **The GRAY row across the bottom contains the eight Common Core State Standards for Mathematical Practice (SMP).**

<table>
<thead>
<tr>
<th>Geometry Topics</th>
<th>Seasons</th>
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<tbody>
<tr>
<td>Essentials of Geometry</td>
<td>FALL</td>
</tr>
<tr>
<td>Congruence, Similarity, and Polygons</td>
<td>WINTER</td>
</tr>
<tr>
<td>Measurement</td>
<td>SPRING</td>
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</table>
**Mathematics | Geometry, Chapter 5**

### SCOPE AND SEQUENCE

<table>
<thead>
<tr>
<th>Topics</th>
<th>Emphasized Standards</th>
<th>Essential Questions</th>
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<tbody>
<tr>
<td><strong>FALL SEASON</strong></td>
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<tr>
<td>Essentials of Geometry</td>
<td><strong>Geometry:</strong> Experiment with transformations in the plane (G-CO 1, 2, 4, 5). Prove theorems about lines and angles (G-CO ‘9’). Make formal geometric constructions (G-CO 12). Use coordinates to prove simple geometric theorems algebraically (G-GPE 5, 6).</td>
<td>How can we use deductive reasoning to reach conclusions about angle measurements or the existence of parallel lines? How do parallel and perpendicular lines help us understand angle relationships? How can you use deductive reasoning to verify perception?</td>
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<tr>
<td>1. Tools of Geometry</td>
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<tr>
<td>2. Reasoning and Proof</td>
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<tr>
<td>3. Parallel and Perpendicular Lines*</td>
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<tr>
<td>4. Coordinate Geometry</td>
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<tr>
<td>5. Transformations</td>
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<tr>
<td><strong>WINTER SEASON</strong></td>
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<tr>
<td>Congruence, Similarity, and Polygons</td>
<td><strong>Geometry:</strong> Experiment with transformations in the plane (G-CO 3). Understand congruence in terms of rigid motions (G-CO 6, 7, 8). Prove geometric theorems (G-CO 10, 11). Understand similarity in terms of similarity transformations (G-SRT-1, 2, 3). Prove theorems involving similarity (G-SRT-4, 5). Define trigonometric ratios and solve problems involving right triangles (G-SRT-6, 7, 8). Use coordinates to prove geometric theorems algebraically (G-GPE 4).</td>
<td>How does the Pythagorean theorem help me solve real-world problems? How does trigonometry relate to triangles and help us solve real-world problems? How can I use trigonometry to measure the heights of very tall structures? How are rigid motions and congruence related?</td>
</tr>
<tr>
<td>1. Congruent Triangles</td>
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<td>2. Properties of Triangles</td>
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<td>3. Proportions and Similarity</td>
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<td>4. Right Triangles and Trigonometry*</td>
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<td>5. Quadrilaterals and Other polygons</td>
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<tr>
<td><strong>SPRING SEASON</strong></td>
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<td></td>
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<tr>
<td>Measurement</td>
<td><strong>Numbers and Quantity:</strong> Reason quantitatively and use units to solve problems (N-Q 3). <strong>Geometry:</strong> Understand and apply theorems about circles (G-C 1, 2, 3). Find arc lengths and areas of circles (G-C 5). Translate between the geometric description and the equation for a conic section (G-GPE 1). Use coordinates to prove simple geometric theorems algebraically (G-GPE 4, 7). Explain volume formulas and use them to solve problems (G-GMD 1, 3, 4). Apply geometric concepts in modeling situations (G-MG 1, 2, 3). <strong>Statistics:</strong> Understand independence and conditional probability and use them to interpret data from simulations or experiments (S-CP 1, 2, 3, 4, 5).</td>
<td>What are various ways to maximize the volume of a box? Why would a business use one shape instead of another for packaging its products? How do we apply the volume formulas to real-life situations?</td>
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<tr>
<td>1. Areas of Polygons and Circles</td>
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<tr>
<td>2. Surface Area and Volume*</td>
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<tr>
<td>3. Circles</td>
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<td></td>
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<tr>
<td>4. Probability and Measurement</td>
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<td></td>
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<tr>
<td>Common Core State Standards for Mathematical Practice (SMP):</td>
<td></td>
<td></td>
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<tr>
<td>1. Make sense of problems and persevere in solving them.</td>
<td>2. Reason abstractly and quantitatively.</td>
<td>3. Construct viable arguments and critique the reasoning of others.</td>
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</tbody>
</table>
### Transfer Goals

<table>
<thead>
<tr>
<th>Application of Learning</th>
<th>Performance Assessment</th>
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</table>
| Students will use their learning to justify their arguments when trying to persuade someone that their line of logic is correct. | Creation of Optical Illusions
Students will create an optical illusion using parallel and perpendicular lines while utilizing appropriate mathematical tools. The illusion will appear not to have parallel lines; however, the student will be able to prove (using two proofs and the theorems discussed in the unit) that the lines are parallel. To create their optical illusions, students might use a compass and straightedge, protractors, paper folding/cutting, or dynamic geometric software. |
| Students will use their learning to accurately describe and model their mathematical skills that occur in the real world. | The Extraordinary Race
Students will be given "The Extraordinary Race" maps and told that they need to complete two parts of this assessment. In Part A, they need to find the quickest way from the drop off point (A) to the end of the race (J). Then, in Part B, students must get to point Z. Each stage of the race requires students to apply their understanding of right triangles and trigonometric ratios to find the fastest route to the end of the race. Students will present the quickest route to the class and explain their reasoning as to why they chose the route that they did. They will use visuals (poster, PowerPoint, storyboard, etc.) to show their classmates how they chose to get through each stage of the race. |
| Students will apply their deductive reasoning skills to determine the accuracy of their perceptions in the world. | Creating Improved Product Packaging
Students will imagine that they are recreating the packaging of a product, and they will pitch their idea in a presentation to executives of the product's company. Students will think about the best packaging that can be created to hold the most product with the smallest surface area. The students will prove to the executives that they will save money by changing the packaging because they will use less material to package the product and will be able to ship it more efficiently. They will compare the new package to the previous package in order to show executives how much packaging material they will save. They will create graphs that show the maximum volume in order to prove that the new package is best. |
| Students will apply their understanding of trigonometry to solve complex problems. | |
| Students will apply their understanding of the concept of proofs to justify their reasoning. | |
| Students will apply their understanding of scale and proportionality to create scale drawings. | |
| Students will apply their understanding of trigonometry to measure the heights of tall or large-scale structures. | |
| Students will apply their understanding of mathematical formulas to calculate the volume of real-world objects. Students will prove that certain shapes work better for different tasks. | |
| Students will apply their understanding of volume and surface area to illustrate various ways to package products and learn why some are more appropriate for use in certain applications than others. | |
| Students will apply their understanding of volume formulas to find the volume of real-world objects. | |

### Common Core State Standards for Mathematical Practice (continued):

| 5. Use appropriate tools strategically. | 6. Attend to precision. | 7. Look for and make use of structure. | 8. Look for and express regularity in repeated reasoning. |
As students begin the Fall Season of geometry, they are introduced to key skills and understandings that will lay the foundation for the rest of this year and future math courses. The Fall Season introduces students to the tools of geometry and prepares them to apply their skills to more complicated problems as the year progresses.

The unit outlined here, “Parallel and Perpendicular Lines,” is designed to come third in the Fall Season of Essentials of Geometry and will take about two weeks to complete. The unit asks students to think about the questions:

- How can we use deductive reasoning to reach conclusions about parallel and perpendicular lines?
- How do parallel and perpendicular lines help us understand angle relationships?

Students will explore these questions throughout the unit by looking at examples of parallel and perpendicular lines, by creating optical illusions for the Performance Task and by proving the existence of parallel lines in those Performance Tasks. The goal of this unit is for students to use their learning to justify their arguments when trying to persuade someone that their line of logic is correct and to use their learning to accurately describe and model their mathematical skills in the real world.

After completing this unit of study, teachers might decide to spend more time discussing the applications of this unit (and the Fall Season in general) to students’ lives and future careers. Teachers could invite carpenters or engineers into the classroom to discuss practical applications of the math that the students are studying.

This unit focuses on standards:

- **G-CO.9**: Prove theorems about lines and angles.
- **G-CO.12**: Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

To meet these standards, students will measure angles and create parallel and perpendicular lines using online tools, protractors, and rulers. They will learn to write proofs and to use deductive reasoning to prove theorems.

To engage students with the Emphasized Standards and Essential Questions of the unit, teachers will ask students to write, read, create, and watch videos about the material that students are learning. Students will be able to work alone, with partners, and with the whole class to solve a variety of problems and to engage in the material. The
final assessment will allow students to create an optical illusion, which students will be able to do on a computer, by drawing, or by cutting and pasting strips onto a larger piece of paper. They will then be asked to prove the existence of parallel lines in their optical illusions since the illusions will appear not to have parallel lines.

Prior knowledge that students need for this unit includes a basic understanding of how to use geometric tools (such as a protractor and ruler). Students should also know how to find the slope of a line and should know that slope = rise over run. If students do not have these prior skills or knowledge, the teacher should plan to spend additional time reviewing this material with students.

The unit may present challenges for some students, especially in the area of writing proofs. Even though students will be introduced to proofs in the unit preceding this one, students likely will have a difficult time explaining their reasoning through the writing of geometric proofs. Teachers will need to explicitly teach students the step-by-step process of proving theorems through the writing of proofs and may decide to spend additional days working with students on this topic.
## Essentials of Geometry: Parallel and Perpendicular Lines

Adapting This Short-Term Unit for Long-Term Programs

### Plan 1 (Short)

**FALL SEASON—Parallel and Perpendicular Lines: Short-Term Programs**

<table>
<thead>
<tr>
<th></th>
<th>MONDAY</th>
<th>TUESDAY</th>
<th>WEDNESDAY</th>
<th>THURSDAY</th>
<th>FRIDAY</th>
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</thead>
<tbody>
<tr>
<td>Week 1</td>
<td><strong>Lesson 1:</strong> Introduction to Parallel and Perpendicular Lines</td>
<td><strong>Lesson 2:</strong> Introduction to the Vocabulary Related to Angles</td>
<td><strong>Lesson 3:</strong> Application of Vocabulary to Angle Measurement</td>
<td><strong>Lesson 4:</strong> Introduction to Proofs</td>
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<tr>
<td>Week 2</td>
<td><strong>Lesson 5:</strong> Practice and Application of Proofs</td>
<td><strong>Lesson 6:</strong> Optical Illusions</td>
<td><strong>Lesson 7:</strong> Creation of Optical Illusions</td>
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### Plan 2 (Long)

**FALL SEASON—Parallel and Perpendicular Lines: Long-Term Programs**

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<tr>
<th></th>
<th>MONDAY</th>
<th>TUESDAY</th>
<th>WEDNESDAY</th>
<th>THURSDAY</th>
<th>FRIDAY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 1</td>
<td><strong>Lesson 1:</strong> Introduction to Parallel and Perpendicular Lines</td>
<td><strong>Review Lesson:</strong> Review of Slope</td>
<td><strong>Lesson 2:</strong> Introduction to the Vocabulary Related to Angles</td>
<td><strong>Lesson 3:</strong> Application of Vocabulary to Angle Measurement</td>
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<tr>
<td>Week 2</td>
<td><strong>Extension:</strong> Solving Angle Measurements and Parallel Lines Using Algebraic Equations</td>
<td><strong>Lesson 4:</strong> Introduction to Proofs</td>
<td><strong>Lesson 5:</strong> Practice and Application of Proofs</td>
<td><strong>Lesson 6:</strong> Optical Illusions</td>
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<tr>
<td>Week 3</td>
<td><strong>Lesson 7:</strong> Creation of Optical Illusions</td>
<td><strong>Unit Extension:</strong> Real-World Connections</td>
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</table>

For a short-term program, the unit plan can stay as is (Plan 1).

To adapt this unit for a longer program, teachers could spend an Extension day after Lesson 3 solving angle measurements and parallel lines by using algebraic equations. An additional Review Lesson can be added to review slope with students, since knowing how to find the slope of a line will be important in this unit. The long-term adaptation also allows for teachers to extend Lesson 5 on proofs since this is something many students struggle with.

For a further Unit Extension, teachers could spend time at the end of the unit showing students how angle measurements and parallel and perpendicular lines apply to careers. Students could research careers such as carpentry and architecture in which it is important for workers to be precise and accurate when using parallel lines and correct angle measurement. With the approval of the Regional or Assistant Regional Education Coordinator and the Program Director, teachers could arrange to invite guest speakers into their classrooms to talk about these careers and why math is important in them, providing students with real-life examples of those using math skills they learned in this unit.
Emphasized Standards (High School Level)

GEOMETRY

G-CO.9: Prove theorems about lines and angles.
Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent, and conversely prove lines are parallel; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.

G-CO.12: Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).
Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.

Essential Questions (Open-ended questions that lead to deeper thinking and understanding)

How can we use deductive reasoning to reach conclusions about the existence of parallel and perpendicular lines?
How do parallel and perpendicular lines help us understand angle relationships?
How can you use deductive reasoning to verify perception?

Transfer Goals (How will students apply their learning to other content and contexts)?

Students will use their learning to justify their arguments when trying to persuade someone that their line of logic is correct.
Students will use their learning to accurately describe and model their mathematical skills in the real world.
Students will apply their deductive reasoning skills to determine the accuracy of their perceptions in the world.

For Empower Your Future Connections, see p. 5.5.1
**Learning and Language Objectives**

By the end of the unit:

KUDs are essential components in planning units and lessons. They provide the standards-based targets for instruction and are linked to assessment.

<table>
<thead>
<tr>
<th>Students should know...</th>
<th>understand...</th>
<th>and be able to...</th>
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</thead>
<tbody>
<tr>
<td><strong>Vocabulary:</strong></td>
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<tr>
<td>Parallel lines</td>
<td>Angles are formed by two parallel lines intersected by a transversal.</td>
<td>Construct an angle bisector and a perpendicular bisector.</td>
</tr>
<tr>
<td>Transversal</td>
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<tr>
<td>Perpendicular</td>
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<tr>
<td>Perpendicular bisector</td>
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<td>Alternate interior angles</td>
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<tr>
<td>Alternate exterior angles</td>
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<tr>
<td>Corresponding angles</td>
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<td>Vertical angles</td>
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<td>Same-side (or consecutive) interior angles</td>
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<td>Linear pairs</td>
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<td>Supplementary angles</td>
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<td>Adjacent angles</td>
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<td>Complementary angles</td>
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</table>

**Properties of parallel lines:**

1. *Parallel lines are in the same plane*
2. *Parallel lines never touch*
3. *If two parallel lines are cut by a transversal, then:*
   - Alternate interior angles are equal in measure
   - Corresponding angles are equal in measure
   - Same side interior angles are supplementary (will equal 180 degrees)

**The properties of parallel lines are connected.**

Theorems and postulates can be proven mathematically.

Identify when two lines are parallel.

Prove properties of parallel lines using deductive reasoning.

Design an optical illusion that uses parallel lines, even though the lines do not appear parallel.
Assessment Evidence

Quality questions raised and tasks
designed to meet the needs of all learners

Performance Task and Summative Assessment (see pp. 5.4.15-5.4.16)
Aligning with Massachusetts standards

Students will create an optical illusion using parallel and perpendicular lines while utilizing appropriate mathematical tools. The illusion will appear not to have parallel lines; however, the student will be able to prove (using two proofs and the theorems discussed in the unit) that the lines are parallel. To create their optical illusions, students might use a compass and straightedge, protractors, paper folding/cutting, or dynamic geometric software.

Pre-Assessment (see p. 5.4.7)
Discovering student prior knowledge and experience
Lesson 1: Do Now: Definitions of parallel and perpendicular lines

Formative Assessments (see pp. 5.4.8-5.4.15)
Monitoring student progress through the unit

Daily Do Now activities
T-chart of types of angles and definitions with pictures
Proof index cards
Lesson 2: Parallel Lines and Transversals (angle measurement) Activity Sheet
Lesson 5: Student-created quiz and answers
Lesson 6: Optical illusion

For Empower Your Future Connections, see p. 5.5.1
Multiple Means of Engagement

This is the *why* of learning. It is what makes students engage or disengage. Throughout the unit plan, the student may be provided with as many choices in the level of challenge and complexity as possible in order to recruit and sustain engagement. For example, the teacher will encourage and support students in setting their own personal, academic, and behavioral goals. The teacher will use many strategies to guide students, including reminders, guides, rubrics, checklists, and prompts among other things that focus students on self-regulatory goals. Student tasks will be varied, allowing for active participation, exploration, and experimentation.

The lessons in this unit are designed to show students the real-world applications of the skills that they are taught. The Performance Task in this unit is designed to allow students to use their knowledge of parallel lines to create an optical illusion that appears to have lines that are not parallel. Also, this task will engage students by asking them to ponder how things in life could appear differently than they actually are. The students will be excited to discover the tricks to optical illusions. More importantly is that teachers design assignments and tasks with authentic outcomes that are purposeful and convey meaning to real audiences.

Students will be expected to “discover” mathematical concepts instead of being told them. They will work independently throughout the unit and with peers in order to learn new concepts, like writing proofs with partners in Lesson 4. In addition, each student will create their own quizzes and exchange their work with another student.

All students are encouraged to demonstrate their proofs, regardless if they come to the “correct” answer. A problem does not have to be done correctly to be worthy of discussion; in fact, it can be helpful to the class to see a problem done incorrectly so that they understand misconceptions.

Multiple Means of Representation

This is the *what* of learning. There are many pathways to conveying information to students. Throughout the unit, the teacher will provide information and materials in several modalities such as diagrams, vocabulary cards, word walls, posters, and charts with formulas for calculations; and models, videos, and audio for text. The teacher will also demonstrate concepts through hands-on activities.

Due to the amount of vocabulary that is associated with this unit, it is important to teach vocabulary in a way that will allow students to connect their prior knowledge to new ideas. They can use T-charts or other organizers to reinforce concepts. They will be expected to find real-life examples of concepts, to draw concepts, and to explain concepts verbally and in writing. The teacher will show videos, pictures, and drawings, as well as use verbal explanations, to explain concepts.

Lessons 1, 2, 6, and 7 have been built with relevant analogies so that students can see real-life examples of concepts. In Lesson 1, students are looking at exemplars of illusions through a PowerPoint. During the
second lesson, students are able to watch a video, “Identifying and Measuring Angles in the Real World,” which permits access to the concepts audibly and visually. If students have difficulty while working on the Performance Task, they may reference the video, “Optical Illusion: Lines of Confusion.”

To support various students’ needs, the students will be able to look at the PowerPoint illusions projected on a screen or have their own personal copy provided to them. Students will also be given the chance to have readings read to them, to read with partners, or to read on their own.

Multiple Means of Action and Expression

This is the how of learning. In the unit’s learning activities, students will be provided options for demonstrating what they know and can do. Students should have multiple means to express their understanding of the concepts presented. For the Performance Task, the teacher can allow students to draw or use a computer to create their optical illusions. Throughout the assignments, students will be able to show their understanding verbally, through writing, and through drawings. During tasks, students can utilize assistive technology, such as word processors or calculators.

There are alternatives that some students might need to physically interact with the materials. For example, some students may have difficulty using a protractor. As an alternative, transparency paper could be offered for construction. Students could also use the website GeoGebra which will find angle measurements for them after they construct parallel and perpendicular lines.

SEE: GeoGebra
app.geogebra.org

Also, in Lesson 6 and 7, the teacher should provide as many tools as possible to support the students’ achieving the learning objective. Tools could include protractors, rulers, or other technical tools.

Some students might have difficulty setting up the proofs in this unit. The teacher should break this down into steps. Students may also express their learning through partnered work and the presentation of their proofs to the class.

Literacy and Numeracy Across Content Areas

Reading

Students will read the instructions carefully in order to complete all assessments. Students will also read and interpret a variety of math texts, such as tables, graphs, and charts in order to solve problems or create solutions. Students will also read about the science behind optical illusions to help them complete the Summative Assessment.

Writing

Students will engage in writing activities through “Do Now” activities or “Exit Tickets.” They will also need to write to express their reasoning and to prove their answers. For the Summative Assessment, students will need to write a short explanation that proves that the lines in their optical illusions are parallel.
Speaking and Listening
Students will speak with their teacher and classmates in order to complete all of the assignments in this unit. Students will share their reasoning with their classmates and build on the ideas of their classmates to clarify their own thinking. For the Performance Task, students will present their illusions to the class. Students will need to listen to their classmates and evaluate their presentations.

Language
Students will use content-specific vocabulary to explain the mathematical concepts discussed in this unit.

Numeracy
Students will explore lines and angle relationships throughout this unit. They will make inferences based on findings of angle measurements and will use angle measurements to prove theorems about parallel lines.

Resources (in order of appearance by type)

Websites
Lesson 1

www.lhup.edu/~dsimanek/3d/illus1.htm

Lesson 2
www.youtube.com/watch?v=SBtojUG1z6s

https://app.geogebra.org/

Lesson 6
www.youtube.com/watch?v=llonA5RtpUM

Lesson 7

Materials
Lesson 1: Parallel and Perpendicular Lines
http://bit.ly/2qp5RVb

Lesson 2: Parallel Lines and Transversals
Activity Sheet p. 5.6.1

Lesson 2: Angle Vocabulary T-Chart
Activity Sheet p. 5.6.2

Lesson 3: Angle and Vocabulary Review Game
Activity Sheet pp. 5.6.3-5.6.4
### PREREQUISITES: Math skills needed for this unit

*Parallel and Perpendicular Lines* is the third unit in the Fall Season of Geometry. The prerequisite math skills summarized below are taught in units that precede this unit (see Scope and Sequence chart). The following skills will be needed for students to successfully complete this unit.

**Students should know:**
- That slope = rise over run
- How to find the slope of a line
- How to use a protractor to measure angles
- The types of angles

### Outline of Lessons

Introductory, Instructional, and Culminating tasks and activities to support achievement of learning objectives

**Note:** Teachers should vet all videos included in this unit according to program standards and create templates or graphic organizers for students to follow along.

### INTRODUCTORY LESSON

*Stimulate interest, assess prior knowledge, connect to new information*

#### Lesson 1

**Introduction to Parallel and Perpendicular Lines**

**Goal**
Students will identify parallel and perpendicular lines and explain why these types of lines matter in our everyday lives.
Do Now (time: 5 minutes)
The teacher will show or project for students the Escher print “Ascension and Descension.”
The teacher will ask students to explain what they see happening here. In what direction are the people moving? Are the stairs leading the people up or down? How can this be? Students will share their answers with the class.

SEE: Ascension and Descension

Hook (time: 5 minutes)
The teacher will tell students that this optical illusion uses parallel lines in order to create an illusion. The teacher will explain to students what parallel lines are. The teacher will show students this excerpt from “Visual Illusions” that explains how parallel lines are used to create optical illusions.

SEE: The Principles of Artistic Illusions
www.lhup.edu/~dsimanek/3d/illus1.htm

Presentation (time: 15 minutes)
The teacher will show students pictures of real-life examples of parallel and perpendicular lines from a PowerPoint (available on Google Drive in the Guide’s supplemental resources for DYS/SEIS teachers).

SEE: Geometry_Parallel_Perpendicular Lines_Lesson 1

Isometric drawings represent all parallel lines as parallel on the flat page, even if they are tilted with respect to the observer in the actual scene. An object tilted away from the observer by some angle looks the same as if it were tilted toward the observer by the same angle. A tilted rectangle has a two-fold ambiguity, as demonstrated by Mach’s figure (left), which may be seen as an open book with pages facing you, or as the covers of a book, with the spine facing you. It may also be seen as two symmetric parallelograms side by side and lying in a plane, but few people describe it that way.

Practice and Application (time: 20 minutes)
The teacher will show students the PowerPoint presentation again and ask students why parallel and perpendicular lines matter.

Why is it necessary to have parallel lines on a football field? On railroad tracks?
What would happen if these lines were not parallel?

As the teacher is going through the PowerPoint, the teacher should ask students:

Do these lines appear to be parallel? How could we tell if they are or are not parallel?
Do our eyes always perceive the “truth”?
What did we notice in the isometric drawing that we just looked at that uses parallel lines?

The teacher will ask students to get with partners to brainstorm as many examples of parallel and perpendicular lines as they can think of and be able to explain why they matter in our everyday lives. Why does it matter that lines are perfectly parallel? If students cannot work with partners, they could brainstorm ideas individually and then share their ideas with the class.

**Review and Assessment** (time: 10 minutes)
Students will share the examples that they discussed with the class. The students should create a class list of examples of parallel and perpendicular lines that they explored. The teacher should provide students with a mathematical definition of parallel and perpendicular lines. Students will create a T-chart with these words, a drawing, and a definition.

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### INSTRUCTIONAL LESSONS

**Lesson 2**

**Introduction to Vocabulary Related to Angles (2 Days)**

**Goal**
Students will discover characteristics of angles that are created by perpendicular and parallel lines.

**Lesson 2—DAY 1:**

**Do Now** (time: 5 minutes)
The teacher will ask students to look around the classroom and find as many different types of angles as possible.

*Note:* While knowing types of angles is a prerequisite to this unit, the teacher might find that the students do not have a good understanding of angles. The teacher might point out one or two angles that she/he notices to get students started with their lists. The teacher will review angles in the hook. This Do Now activity should give the teacher a good understanding of how much students remember about angles.

**Hook** (time: 10 minutes)
Students will share their discoveries with the class. The teacher should use (and review as necessary) the terms *right angle*, *obtuse angle*, and *acute angle* with students. Students will add these terms to their T-charts with definitions and drawings. The teacher will then show students real-life examples of angles and ask them to think about why precise angle measurement is necessary. An option for the teacher might be to use this short video on identifying and measuring angles.

**SEE:** Identifying and measuring Angles in the Real World
www.youtube.com/watch?v=SBtojUG1z6s
Note: The video reviews the types of angles, then at 55 seconds shows students examples of types of angles and gives them about five seconds to classify the angle before giving students the answer.

Presentation (time: 10 minutes)
The teacher will give students examples of parallel lines that are bisected by a transversal. The teacher can use the “Parallel Lines and Transversals” Activity Sheet on p. 5.6.1 in the Supplement, or create his/her own. The teacher will tell students to label the angles, measure the angles, and note what they discover. The teacher should demonstrate how to use a protractor to measure angles if the students don’t remember how to use one. The teacher will ask the students:

Are there any relationships between the angles?

Practice and Application (time: 30 minutes)
With partners, students will label the angles and use protractors to measure all of the angles on the handout. After measuring all of the angles, the students should discuss anything that they notice with their partners. They will create a list of observations that they will share with the class. An alternative or extension would be for students to use the website:

SEE: GeoGebra
https://app.geogebra.org

This website allows students to create parallel and perpendicular lines. The website will find angle measurements for the students. They can explore connections between angles in this way, too.

Lesson 2—DAY 2:
Review and Assessment (time: 55 minutes)
Students will share their observations with the class. They should have made observations that some angles have the same measurement and that angles that are next to each other equal 180 degrees. Students may not have the vocabulary to explain this mathematically, but they should do the best that they can to explain what they discovered.

When students are sharing these observations, the teacher will confirm the students’ findings by using accurate vocabulary. For example, if students say that angles that are next to each other equal 180 degrees, the teacher would say that is correct because supplementary angles equal 180 degrees. The teacher could ask students to list all examples of supplementary angles on the handout.

The vocabulary that should be reviewed with students includes:

*Perpendicular bisector, angle bisector, transversal, alternate interior angles, alternate exterior angles, corresponding angles, vertical angles, same-side (or consecutive) interior angles, linear pairs, supplementary angles, adjacent angles, and complementary angles*

The teacher should also introduce the terms *corresponding angles postulate* and *parallel lines postulate*.

Students should make a T-chart of this new vocabulary and definitions (see the “Angle Vocabulary T-Chart” Activity Sheet located in the Supplement on p. 5.6.2). Students should add a small drawing to each definition. The teacher should collect these T-charts as Exit Slips to confirm whether students understand the vocabulary.
Lesson 3

Application of Vocabulary to Angle Measurement

Goal
Students will apply their knowledge of appropriate vocabulary to identify and practice measuring angles formed by parallel and perpendicular lines.

Do Now (time: 5 minutes)
When students walk into the room, the teacher should have a pair of parallel lines bisected by a transversal on the board. One angle should be marked as 60 degrees. The teacher will ask students to find the measurement of all other angles. The teacher should hand back the T-charts that were collected the previous day.

Hook (time: 5 minutes)
Students will share the answers to the Do Now activity while using appropriate vocabulary. Students can use the T-charts that they made the day before as a reference to ensure that they are using correct vocabulary.

Presentation (time: 10 minutes)
The teacher will review the vocabulary and clarify any misconceptions. The teacher can do this as s/he is posting a word wall of vocabulary from yesterday’s lesson. Since the teacher collected the T-charts the day before, he or she should know how much review is necessary for students to be successful today.

Practice and Application (time: 25 minutes)
Students will play a review game in teams that will ask them to use appropriate vocabulary and to find angle measurements. The teacher will divide the class into two teams. The members of each team will work together to come up with the correct answer to questions that the teacher poses. When the team thinks that it has the correct answer, a designated team member will raise his/her hand to answer the question. If the team gets the answer correct, the team wins a point. If the team answers the question incorrectly, the other team has a chance to answer the question.

The teacher will use practice materials from the Supplement as a resource to create questions for the game (see the “Angle and Vocabulary Review Game” Activity Sheet on pp. 5.6.3-5.6.4). For each of the six examples on the sheet, the teacher can ask multiple questions. The teacher needs to reinforce the appropriate vocabulary throughout the lesson so that students are familiar with it.

Review and Assessment (time: 10 minutes)
As an Exit Ticket, each student will create two questions (with an answer key) that illustrate what they have learned about parallel lines, perpendicular lines, and angle measurements.

Extension
As an extension of this lesson, the teacher may choose to review algebraic expressions instead of angle values while asking students to find angle measurement.
Lesson 4
Introduction to Proofs

Goal
Students will use deductive reasoning and logic to prove the properties of parallel lines. Students will write two-column proofs.

Do Now (time: 5 minutes)
The teacher will write the following prompt on the board:

If two parallel lines are cut by a transversal, then…

The teacher will ask students to list (in a notebook or on paper) everything they know and have learned so far that completes this sentence. For example, alternate interior angles are equal, vertical angles are equal, and so on.

Hook (time: 5 minutes)
The teacher will draw a picture like the one below on the board. The students should assume that angles A, D, E, and H are 120° (degrees) and ask students to use deductive reasoning to determine if the two horizontal lines are parallel:

\[\begin{align*}
\text{A} & \quad \text{B} \\
\text{C} & \quad \text{D} \\
\text{E} & \quad \text{F} \\
\text{G} & \quad \text{H}
\end{align*}\]

The teacher will review Do Now activity answers and the responses to the Hook with students and point out the theorems and postulates that they learned in Lesson 2, making sure to add any information that students might have missed. The students should know the following information:

If two parallel lines are cut by a transversal, the corresponding angles are congruent; and if two parallel lines are cut by a transversal, the alternate interior angles are congruent.

The teacher can review supplementary and complementary angles with students to ensure their understanding of the theorems and postulates that they already learned. The teacher will ask students:

How do we know these are true?
How can we confirm beyond the teacher telling you that these statements are true?
Presentation (time: 30 minutes)
The teacher should have each theorem and postulate printed on an index card. A copy of these theorems and postulates can be found on the “Parallel and Perpendicular Lines—Theorems, and Postulates” Activity Sheet on p. 5.6.5 in the Supplement. The teacher will pair students up and ask them to pick a card. Each pair will have 10 minutes to use everything they know about parallel and perpendicular lines, and about math in general, to prove that their theorem/postulate is indeed true.

Each pair will present their “proof” to the rest of the class with the teacher supporting and making corrections where needed.

After all the pairs have presented, the teacher will introduce the two-column proof format to students using some of their examples, as well as at least one new one.

Practice and Application (time: 10 minutes)
Each student pair will select a new index card and use the two-column proof format to prove their new theorem/postulate.

Review and Assessment (time: 5 minutes)
Students will ask any questions that they have. They will complete a 3-2-1 Exit Slip created by themselves or the teacher that asks for three things they learned so far in the unit, two questions they have, and one skill that they can now do. Students will turn in their proofs at the end of class.

Lesson 5
Practice and Application of Proofs

Goal
Students will review guidelines for writing two-column proofs. Students will review facts regarding parallel and perpendicular line properties and how they support writing proofs. Students will practice writing proofs.

Do Now (time: 10 minutes)
The teacher will write several proof problems on the board (one for each student in the class), as well as the headings of the two-column proof format, and will invite students to step up to the board to solve them.

After about 3–4 minutes of students working on their proofs (they’ve written a line or two), the teacher will ask students to stop and switch places with the person next to them and complete their work.

Note: The teacher can choose to do a couple of rounds of this, asking students to switch to another proof multiple times. For students who may feel “territorial” about their work, the teacher may tell them to put their initials next to the lines they’ve completed. The teacher could have plastic sleeves to put in the student work and have other students write with dry erase on the sleeve to protect student work. Another option is for the teacher to use an Eno® interactive whiteboard or other projection board to show the proof problems while students, individually or in pairs, solve the problems at their seats.
Hook (time: 5 minutes)
The teacher will review each proof with the class, checking for understanding and allowing students to correct any mistakes found.

Presentation (time: 20 minutes)
The teacher will review all of the postulates and theorems (See p. 5.6.5, “Parallel and Perpendicular Lines—Theorems and Postulates” in the Supplement) that students proved the day before and will ask students to continue working in pairs to prove any postulates and theorems that weren’t done the previous day. The goal is to show students that all theorems and postulates can be (and should be) proven. Pairs will present their proofs as they did the day before.

Practice and Application (time: 10 minutes)
The teacher will ask students to create 5-question quizzes with answer keys about parallel and perpendicular lines, their properties, and proofs. At least one question should be about creating a proof. The teacher should circulate around the room as the students are working on the “quiz” in order to answer any questions they may have.

Review and Assessment (time: 10 minutes)
Students will trade quizzes with their partners and answer the questions that their partners created. If students are having trouble answering a question, they can ask partners for help. The students should turn in their quizzes to the teacher at the end of class. The teacher should review both the questions created and the student answers to ensure understanding.

CULMINATING LESSONS
Includes the Performance Task, i.e., Summative Assessment—measuring the achievement of learning objectives

Lesson 6
Optical Illusions

Goal
Students will prove that lines in optical illusions are parallel.

Do Now (time: 5 minutes)
The teacher will ask the students to respond to the following question in writing or by discussing with partners:

How do we know when we are looking at parallel lines?

Hook (time: 10 minutes)
The teacher will use the Optical Illusions Powerpoint to project the image that appears on p. 5.4.15:

SEE: Optical Illusions PowerPoint
The teacher will ask students:

Are these lines parallel?
How do we know?

Students should discuss their thoughts with partners before sharing their thoughts with the class.

Presentation (time: 5 minutes)
The teacher will continue to show a series of optical illusions to the class. See the PowerPoint referenced on p. 5.4.6 and located in the Google Drive resource folder.

SEE: Optical Illusions PowerPoint

All of these pictures use parallel lines, but our eyes do not perceive them to be parallel. The teacher will divide students into pairs and will give each pair a print-out (from the PowerPoint) of an optical illusion. The teacher will tell students that they need to decide if the lines are parallel and prove whether or not they are.

Practice and Application (time: 25 minutes)
Students will study the illusions they are given and decide how to prove that the lines in their illusion are parallel. They will use their knowledge of parallel lines, protractors, rulers, or any other tools at their disposal to prove whether or not the lines are parallel. The teacher should circulate around the room to help students who are struggling and to monitor the conversations that are taking place about parallel lines.

Review and Assessment (time: 10 minutes)
Pairs of students will present to the class how they proved that their optical illusion actually uses parallel lines. Using the theorems that they have learned, students should be able to explain and prove that the lines in their optical illusions are parallel.
Lesson 7

Creation of Optical Illusions (2 Days)

On Day 1, the teacher will lead students through activities leading up to Practice and Application, and students will begin creating their optical illusions. On Day 2, students will complete their optical illusions and present them to the class.

Goal
Students will create an optical illusion that uses parallel and perpendicular lines. After creating the optical illusion, they will prove that the optical illusion has parallel lines using the theorems learned in this unit. Students will use two different proofs to prove that their lines are parallel.

Do Now (time: 10 minutes)
The students will complete a quick write responding to the following prompt:

What did you learn yesterday about optical illusions?

Students will discuss their thoughts with a partner before sharing them with the class.

Hook (time: 10 minutes)
The teacher will share the article “How Do Optical Illusions Work?” with students.

SEE: How Do Optical Illusions Work?

The teacher will read the article out loud to students or have students read the article with partners. Students will complete a T-chart or other graphic organizer to identify the main idea and summarize the article.

Presentation (time: 10 minutes)
The teacher will challenge students to create an optical illusion of their own. The students must use parallel lines in their optical illusion and be able to prove that the lines are parallel. The challenge is that students will want to make their optical illusion appear not to have parallel lines.

The teacher will create a rubric with the class to assess students’ optical illusions. The teacher and students should decide on the criteria to include in the rubric. The teacher will want to remind students that while they might want to create a colorful illusion, the point of this assignment is not simply to create a work of art. The important piece of this project is that the students use their mathematical skills and reasoning to prove that the lines in their drawing are parallel. This needs to be reflected in the rubric. They also need to use tools such as a compass and straightedge, protractors, or the website below in order to create this project and prove that the lines are parallel.

SEE: GeoGebra
https://app.geogebra.org

Practice and Application (time: 60 minutes | Day 1—25 minutes, Day 2—35 minutes)
Students will begin creating their optical illusions on Day 1 (time: 25 minutes) and continue the work on Day 2 (time: 25 minutes). The teacher should monitor the work that students are doing throughout the activity. This will likely take students a lot of time to discover how to make their parallel lines appear not parallel, so the teacher should be patient as students use trial and error to make their illusions. The teacher should
provide students with rulers, protractors, and colored pencils or markers so that students can create these illusions.

If students are having a lot of difficulty creating their illusions, the teacher can allow students to do some research on how to create optical illusions using parallel lines or show students the following video.

**SEE:** Optical Illusion: Lines of Confusion
www.youtube.com/watch?v=llonA5RtpUM

Once students have created their optical illusions, they should write brief explanations to prove that the lines are parallel. They should use two different proofs that prove that the lines are parallel.

**Review and Assessment** (time: 35 minutes)
Students will present their optical illusions to the class. They will explain the two ways that they can prove that their lines are actually parallel, even though they appear not to be. They will explain which tools they used to prove that the lines are parallel (protractors or online tools such as GeoGebra if the illusion was created on a computer) and how they used the tools in order to mathematically prove that the lines are parallel.

**SEE:** GeoGebra
https://app.geogebra.org

**POST–UNIT REFLECTION**
*On meeting the Learning and Language objectives*
Connections to Empower Your Future
UNIT: Parallel and Perpendicular Lines

Future Ready Connections

*The skills needed to demonstrate the Emphasized Standard G-CO.12 connect to Future Ready skills.* The standard assesses students’ ability to make geometric constructions with a variety of tools and methods that provide teachers with the opportunity to monitor and assess students’ initiative and self-direction. Youth will have to experiment with the different methods for measuring angles, create intersecting lines, and then reflect on the use of each method. Students must take the initiative to experiment, reflect, and come to conclusions based on their experiments. This process requires critical thinking skills, analysis, and evaluation which are essential aspects of Future Ready preparation.

*Teachers are encouraged to use the Future Ready Rubric* to evaluate students and are encouraged to support students as they self-evaluate their demonstration of Future Ready skills. Teachers are encouraged to specifically name the workplace readiness skills students will use while completing their projects and ask students to reflect on how they demonstrated those skills, what they could do to further improve them, and how they are transferable to other situations and experiences.

Essential Questions

One of the Essential Questions asks youth to consider how they can use deductive reasoning to reach conclusions about parallel and perpendicular lines. Teachers can ask youth to brainstorm times when they use deductive reasoning outside of the classroom. Teachers can start with the example “If \( A = B \) and \( B = C \), then \( A = C \)” and then ask students to brainstorm everyday examples such as “It is dangerous to drive on icy streets. It is icy now. Therefore it is dangerous to drive” or “All dolphins are mammals, and all mammals have kidneys; therefore all dolphins have kidneys.”

Students can also reflect on how this type of reasoning compares to inductive reasoning and discuss why deductive reasoning is appropriate and necessary for mathematics. An example of inductive reasoning is “All living things that we know of depend on liquid water to exist. Therefore, if we discover a new living thing, it will depend on liquid water to exist.” An example of faulty inductive reasoning is “All of the swans we have seen are white. Therefore, all swans are white.” Students can discuss different examples of deductive and inductive reasoning and apply it to mathematical concepts to determine if the logic is correct or faulty.

The skill of using deductive reasoning is an essential aspect of critical thinking and of many career

Students can brainstorm career fields, jobs, or even tasks at a job that will require deductive reasoning. Some examples include doctors, police officers, athletic trainers, coaches; planning a supply order, calculating size/weight of a shipment, etc. It is important to demonstrate to youth that they likely already use deductive reasoning on a daily basis and that further developing this skill will be useful in both their personal and professional lives.

Transfer Goal Connections

One of the Transfer Goals states that students will be able to use their learning to defend their arguments when attempting to persuade someone that their line of logic is correct. Teachers are encouraged to assess this skill using the Effective Communication section on the
“The skill of using deductive reasoning is an essential aspect of critical thinking and of many career fields.”

Future Ready Rubric. Teachers can expand on this idea of persuasion and effective communication by introducing students to the three types of persuasion: ethos (being a trusted voice on the subject), pathos (appealing to the listener’s emotions), logos (logical and factual support for an argument).

Students can apply their persuasive skills to other subjects and topics and can also identify career fields that depend on strong persuasive and oral skills such as politics, law, academia, or sales.

PYD/CRP Connections

Many lessons use real-life examples of parallel and perpendicular lines and ask youth to identify additional examples from their own experience and observations (Lesson 2–Hook, Lesson 6). This reinforces the idea that mathematics is an important part of our daily lives and that students can and should draw from their own experience to further their understanding.

The Emphasized Standard G-CO.12 also reflects Culturally Responsive Practice and Positive Youth Development because students are able to explore a variety of tools and practice different skills and methods. Youth are able to depend on their strengths and prior knowledge to complete the math problems while also getting the chance to challenge themselves to develop new skills and learn how to use new tools.

Lesson 5 encourages Positive Youth Development in the Practice and Application and the Review and Assessment sections. Students will create a 10-question quiz that their partners will need to complete. Students will provide an answer key and will be responsible for each other’s learning. By allowing youth to take the responsibility for their own and each other’s learning, teachers can emphasize Positive Youth Development as students will be able to see themselves as both the learner and the teacher. Teachers should consider having youth reflect on the process of creating the quiz and supporting their partners and ask them to think about what skills they used and what this activity shows about their own understanding.

Lesson 6 includes optical illusions and gives youth the opportunity to analyze and create their own optical illusions. Students are likely familiar with optical illusions, and this connection to their own experience shows Culturally Responsive Practice. Allowing students to experiment and create their own illusions also demonstrates Positive Youth Development as youth are able to apply what they have learned to create their own interesting illusion that they can analyze and explain the use of mathematical concepts.

For Technical Assistance with Empower Your Future connections and lessons, please request support by submitting a Coaching Request ticket using the Coaching Feature on TeachPoint.
Parallel Lines and Transversals
Lesson 2

DIRECTIONS:
1. Find and label as many angles as you can. Use numbers or letters to label each angle.
2. Use a PROTRACTOR to measure all the angles.
3. Make a list of your observations about the angles and their relationships.

OBSERVATIONS
Angle Vocabulary T-Chart
Lesson 2

**DIRECTIONS:** Complete the chart.

<table>
<thead>
<tr>
<th>Angle Type</th>
<th>Definition and Drawing</th>
</tr>
</thead>
<tbody>
<tr>
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</tbody>
</table>
Angle and Vocabulary Review Game
Lesson 3

DIRECTIONS: Use the pictures below to answer the questions that your teacher poses.

1. \( X = \) \(102^\circ\) Angle Type __________________
2. \( X = \) \(85^\circ\) Angle Type __________________
3. \( X = \) \(82^\circ\) Angle Type __________________
4. \( X = \) \(76^\circ\) Angle Type __________________
5. \( X = \) \(133^\circ\) Angle Type __________________
6. \( X = \) \(54^\circ\) Angle Type __________________

Adapted from Handouts produced by www.mathworksheets4kids.com
Angle and Vocabulary Review Game
Lesson 3

Use the information below to answer the questions that your teacher poses.

\[
\begin{align*}
X &= 95^\circ \quad \text{Angle Type } \quad \quad \quad \quad \quad \quad \quad X &= 102^\circ \quad \text{Angle Type } \\
X &= 104^\circ \quad \text{Angle Type } \quad \quad \quad \quad \quad \quad \quad X &= 98^\circ \quad \text{Angle Type } \\
X &= 126^\circ \quad \text{Angle Type } \quad \quad \quad \quad \quad \quad \quad X &= 133^\circ \quad \text{Angle Type }
\end{align*}
\]
## Parallel and Perpendicular Lines—Theorems and Postulates

### Lessons 4 and 5

### Theorem 1.6.1:
If two lines are perpendicular, then they meet to form right angles.

![Diagram of perpendicular lines forming a right angle]

### Corresponding Angles Postulate, or CA Postulate:
If two parallel lines are cut by a transversal, then corresponding angles are congruent.

<table>
<thead>
<tr>
<th>A (\parallel) B</th>
<th>(C \parallel D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E \parallel F)</td>
<td>(G \parallel H)</td>
</tr>
</tbody>
</table>

Equality of Corresponding Angles
\(A=E, C=G, B=F, D=H\)

### Theorem 1.7.1:
If two lines meet to form a right angle, then these lines are perpendicular.

![Diagram of perpendicular lines]

### Alternate Interior Angles Theorem, or AIA Theorem:
If two parallel lines are cut by a transversal, then alternate interior angles are congruent.

\[\angle A = \angle D\]
\[\angle B = \angle C\]

### Theorem 1.7.2:
If two angles are complementary to the same angle (or to congruent angles) then these angles are congruent.

1 is comp to 3
\[\angle 1 \cong \angle 3\]
\[\angle 2 \cong \angle 3\]
\[\angle 1 \cong \angle 2\]

### Alternate Exterior Angles Theorem, or AEA Theorem:
If two parallel lines are cut by a transversal, then alternate exterior angles are congruent.

\[\angle A = \angle D\]
\[\angle B = \angle C\]

### Theorem 1.7.3:
If two angles are supplementary to the same angle (or to congruent angles), then the angles are congruent.

\[\angle 1 \cong \angle 3\]
\[\angle 2 \cong \angle 3\]
\[\angle 1 \cong \angle 2\]

### Consecutive Interior Angles Theorem:
If two parallel lines are cut by a transversal, then alternate interior angles are supplementary.

\[A+C=180, B+D=180\]

### Linear Pair Postulate:
If two angles form a linear pair, then the measures of the angles add up to 180°.

\[A+B = 180\]

### Consecutive Exterior Angles Theorem:
If two parallel lines are cut by a transversal, then alternate exterior angles are supplementary.

\[A+C=180, B+D=180\]

### Vertical Angles Postulate:
If two angles are vertical angles, then they are congruent (they have equal measures).

\[\text{Angle } A = \text{Angle } B\]

### Note: \(\cong\) is the symbol for congruence.
Optical Illusion
Lesson 6
**Right Triangles and Trigonometry Ratios**

**TOPIC SEASON | Geometry—Congruence, Similarity, and Polygons**

This unit is designed for use in long-term programs, but can be adapted to short-term settings

Unit Designers: R. Dubuisson and K. Miele

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**Introduction**

As students begin their study of right triangles and trigonometry, they might be interested to know that the study of right triangles and the Pythagorean theorem have been around since at least 500 BC, when the Greek mathematician Pythagoras lived. Even though the theorem was named after Pythagoras, there is evidence that nearly 1,000 years before Pythagoras lived, the ancient Babylonians had an understanding of the importance of right triangles and the relationships between the sides of those triangles.

“Right Triangles and Trigonometry Ratios,” the fourth unit in the Winter Season of *Congruence, Similarity, and Polygons*, will take students three weeks to complete. In order to engage in the mathematical issues of this unit, students will explore the questions:

- How does the Pythagorean theorem help me solve real-world problems?
- How does trigonometry relate to triangles and help us solve real-world problems?
- How can I use trigonometry to measure the height of very tall structures?

Whenever possible, the teacher should ensure that students are thinking about the skills and knowledge under study through the lens of issues that they might encounter in their daily lives. For example, students will use the Pythagorean theorem to find the distance between points and will explore how the Pythagorean theorem is used in video game design, in measurement, and in GPS navigation. Various readings, games, and videos have been included in the unit in order to ensure that students are engaging with the material in a variety of ways.

This unit focuses on standards:

- **G-SRT.4**: Prove theorems about triangles
- **G-SRT.5**: Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures
- **G-SRT.6**: Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles
- **G-SRT.7**: Explain and use the relationship between the sine and cosine of complementary angles
- **G-SRT.8**: Use trigonometric ratios and the Pythagorean theorem to solve right triangles in applied problems.

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“How does the Pythagorean theorem help me solve real-world problems?”
In order to engage with these standards, students will participate in a variety of activities that ask them to discover and prove the Pythagorean theorem. Students will participate in hands-on activities, such as building a clinometer, in order to measure the height of tall objects that could not easily be measured otherwise.

The Performance Task for this unit asks students to use their knowledge of right triangles and trigonometry to make their way through an imaginary obstacle course in the shortest amount of time. Students will have to support the route that they choose with mathematical reasoning and then present their findings to the class. This task requires that students synthesize all of the skills and knowledge that they have gained throughout the unit and will give teachers a clear understanding of whether or not the students can apply their learning to a new task.

Before studying this unit on right triangles and trigonometry, students have spent many weeks studying congruent triangles, properties of triangles, and proportions and similarity. Students will need to have prerequisite skills from the earlier units in this season, as well as from other math courses that they have taken. They must be able to solve algebraic expressions with radicals, know how to factor and find multiples of numbers, and know how to identify congruent triangles and their postulates. Teachers can find supplemental material in the Lesson 2 extension activity if students need additional practice simplifying radicals.

For adaptation ideas for this unit, see p. 5.7.3 on the right.
### Congruence, Similarity, and Polygons: Right Triangles and Trigonometry Ratios

Adapting This Long-Term Unit for Short-Term Programs

#### Plan 1 (Long)

**WINTER SEASON—Right Triangles and Trigonometry Ratios: Long-Term Programs**

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<tr>
<td>Week 1</td>
<td><strong>Lesson 1:</strong> Introduction to the Pythagorean Theorem</td>
<td><strong>Lesson 2:</strong> Practice Using the Pythagorean Theorem/Simplifying Radical Expressions</td>
<td><strong>Lesson 3:</strong> Proof of the Pythagorean Theorem</td>
<td><strong>Lesson 4:</strong> The Converse of the Pythagorean Theorem</td>
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<tr>
<td>Week 2</td>
<td><strong>Lesson 6:</strong> Similar Right Triangles</td>
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<td><strong>Lesson 8:</strong> Introducing Trigonometry</td>
<td><strong>Lesson 9:</strong> Trigonometric Ratios</td>
<td><strong>Lesson 10:</strong> Clinometers and Ceiling Height (1)</td>
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<td>Week 3</td>
<td><strong>Lesson 11:</strong> Clinometers and Ceiling Height (2)</td>
<td><strong>Lesson 12:</strong> Extraordinary Race</td>
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#### Plan 2 (Short)

**WINTER SEASON—Right Triangles and Trigonometry Ratios: Short-Term Programs**

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</tr>
</thead>
<tbody>
<tr>
<td>Week 1</td>
<td><strong>Lessons 1/4:</strong> Introduction to the Pythagorean Theorem/Converse</td>
<td><strong>Lessons 2/3:</strong> Solving Radicals and Proof of the Pythagorean Theorem</td>
<td><strong>Lesson 5:</strong> The Distance Formula</td>
<td><strong>Lesson 6:</strong> Similar Right Triangles</td>
<td><strong>Lesson 7:</strong> Special Right Triangles</td>
</tr>
<tr>
<td>Week 2</td>
<td><strong>Lesson 8:</strong> Introducing Trigonometry</td>
<td><strong>Lesson 9:</strong> Trigonometric Ratios</td>
<td><strong>Lessons 10/11:</strong> Clinometers and Ceiling Height (1-2)</td>
<td><strong>Lesson 12:</strong> Extraordinary Race</td>
<td><strong>Lesson 13:</strong> Extraordinary Race Presentations</td>
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</table>

In the long-term plan (Plan 1), the unit stays as presented.

The short-term adaptation reduces a lot of the practice time included in the long-term format. The teacher would present **both the Pythagorean theorem and its converse** on the first day as a way to introduce the unit and the formula that students will use to solve side lengths of triangles. While the adaptation does not provide much time to review how to solve for radicals, teachers could review radical expressions in the Do Now examples in subsequent lessons as a means of continuing to review the skill throughout the unit.

For the short-term adaption, the ceiling height lab and the number of days allocated to the Performance Task have also been reduced. For **Clinometers and Ceiling Height**, the teacher could build clinometers for the students instead of asking them to build their own. As the **Extraordinary Race** Performance Task is shortened, students should be focused on explaining the math involved in the task instead of re-creating the map or the visual to go with the Performance Task. They might not have time to do a “final version” of a presentation, but could create a draft that could be presented. The focus of the Performance Task, regardless of how much time is given to create it, should be on the mathematical skills being assessed and the students’ abilities to explain their reasoning.
Emphasized Standards *(High School Level)*

**GEOMETRY**

**G-SRT.4:** Prove theorems about triangles.
*Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely, the Pythagorean theorem proved using triangle similarity.*

**G-SRT.5:** Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

**G-SRT.6:** Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

**G-SRT.7:** Explain and use the relationship between the sine and cosine of complementary angles.

**G-SRT.8:** Use trigonometric ratios and the Pythagorean theorem to solve right triangles in applied problems.

**Essential Questions** *(Open-ended questions that lead to deeper thinking and understanding)*

How does the Pythagorean theorem help solve real-world problems?

How does trigonometry relate to triangles and help solve real-world problems?

How can trigonometry be used to measure the height of very tall structures?

**Transfer Goals** *(How will students apply their learning to other content and contexts?)*

Students will apply their understanding of trigonometry to solve complex problems.

Students will apply their understanding of the concept of proofs to justify their reasoning.

Students will apply their understanding of trigonometry to measure the heights of tall or large-scale structures.
# Learning and Language Objectives

By the end of the unit:

KUDs are essential components in planning units and lessons. They provide the standards-based targets for instruction and are linked to assessment.

<table>
<thead>
<tr>
<th>Students should know...</th>
<th>understand...</th>
<th>and be able to...</th>
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</table>
| **The Pythagorean theorem**  
\(A^2 + B^2 = C^2\)  
Square, square roots  
**Pythagorean theorem converse**  
(If \(A^2 + B^2\) does not equal \(C^2\), then the triangle is not a right triangle) | The Pythagorean theorem can be used to solve real-world problems and to calculate distances between points. | Solve problems that students may encounter in real-life situations by using the Pythagorean theorem. |
| **Similarity**  
**Ratios**  
**Proportions** | The similarities of triangles are based on proportionality relationships. | Prove that triangles are similar by showing that they have congruent corresponding angles and congruent sides.  
Solve for missing sides of similar right triangles. |
| **Special right triangles** | The legs and hypotenuse of certain right triangles have distinct relationships. | Derive the formulas for the lengths of special right triangles. |
| **Sine**  
**Cosine**  
**Tangent** | The height of tall objects can be measured by using mathematical equations. | Measure the height of tall objects using a clinometer and mathematical equations.  
Solve for lengths of sides of triangles using trigonometric ratios. |
Assessment Evidence
Quality questions raised and tasks
designed to meet the needs of all learners

Performance Task and Summative Assessment (see pp. 5.8.23-5.8.25)
Aligning with Massachusetts standards
Students will be given “The Extraordinary Race” maps and told that they need to complete two parts of this assessment. First, in Part A, they need to find the quickest way from the drop off point (A) to the end of the race (J). Then, in Part B, students must get to point Z. Each stage of the race requires students to apply their understanding of right triangles and trigonometric ratios to find the fastest route to the end of the race. Students will present the quickest route to the class and explain their reasoning as to why they chose the route that they did. They will use visuals (poster, PowerPoint, storyboard, etc.) to show their classmates how they chose to get through each stage of the race.

Pre-Assessments (see pp. 5.8.9-5.8.12)
Discovering student prior knowledge and experience
Lesson 1: Right Triangle Pre-Assessment
Lessons 1/2: Task Cards

Formative Assessments (see pp. 5.8.10-5.8.23)
Monitoring student progress through the unit
Daily Do Nows and Exit Tickets.
Lesson 2: Radicals Extension Worksheet
Lesson 2: Pythagorean Theorem Word Problems
Lesson 3: Proof of the Pythagorean Theorem Lab
Lesson 4: Converse of the Pythagorean Theorem
Lesson 5: Distance Formula Activity
Lesson 6: Similar Right Triangle Activities 1 and 2
Lesson 6: Eyes and Ears Protocol
Lesson 7: Special Right Triangle Investigations 1 and 2
Lesson 8: Discovering Trigonometry Activity
Lesson 9: Right Triangle Ratios Lab
Lessons 10/11: Ceiling Height Estimation Lab

For Empower Your Future Connections, see p. 5.9.1
Multiple Means of Engagement

This is the *why* of learning. It is what makes students engage or disengage. Throughout the unit plan, the student will be provided with as many choices in the level of challenge and complexity as possible in order to recruit and sustain engagement. For example, the teacher will encourage and support students in setting their own personal, academic, and behavioral goals. The teacher will use many strategies to guide students, including reminders, guides, rubrics, checklists, and prompts among other things that focus students on self-regulatory goals. Student tasks will be varied, allowing for active participation, exploration, and experimentation. The teacher will provide differentiated models, scaffolds, and feedback, as well as content material that is culturally relevant and responsive to students’ backgrounds. Most important is that teachers design assignments and tasks with authentic outcomes that are purposeful and convey meaning to real audiences.

The lessons in this unit are designed so students can use the discovery process to learn mathematical skills and concepts. The first lesson asks students to discover the Pythagorean theorem before they are shown the equation and how to use it. Students will also be given examples of how to use mathematical concepts and skills in real-world situations, e.g., how can I find the shortest distance or the quickest route between two points on a map and know how far I have traveled? Students will also explore careers that use right triangles and the Pythagorean theorem. Students will build clinometers to be able to measure the heights of objects that are too tall to measure in other ways. Evaluative emphasis should be placed on process, effort, and improvement. Formative Assessments are designed to invite personal response, self-evaluation, and reflection. Students should be given problems and be shown examples that address a wide range of diversity and learning profiles in the classroom. Where possible, work should be done in pairs and/or small heterogeneous cooperative learning groups. The Performance Task is divided into many stages to help students tackle one section of the map at a time. Visuals are also included so students can see the terrain described on the map and the challenges faced along the way.

Modifications intended to adjust the unit’s Learning and Language Objectives, Transfer Goals, level of performance and/or content will be necessary for students with mandated specially designed instruction described in their Individualized Education Programs (IEPs).

Multiple Means of Representation

This is the *what* of learning. There are many pathways to conveying information to students. Throughout the unit, the teacher will provide information and materials in several modalities such as diagrams, vocabulary cards, word walls, posters, and charts with formulas for calculations; and models, videos, and audio for text. The teacher will also demonstrate concepts through hands-on activities such as the building of clinometers.

The way information is displayed should vary, including size of text, images, graphs, tables, or other visual content. Where possible, written transcripts for videos and auditory content should be provided. Teachers
will use videos to show students real-world connections to the mathematical skills they are studying. Students will engage with the videos by taking notes and filling out the “Eyes and Ears” protocol. Videos can also be used to show students how to build and use clinometers. Information should be chunked into smaller elements, and complexity of questions can be adjusted based on prior knowledge competency. Reference sheets for examples, notes, vocabulary, and definitions can be differentiated for content.

**Multiple Means of Action and Expression**

This is the *how* of learning. In learning activities, students will be provided options for demonstrating what they know and can do. Students will have options of writing or verbalizing their learning. Students will have access to assistive technology. For example, students will have access to word processors with grammar checks, word prediction, and spell checkers. The teacher will also break down long-term goals into short-term reachable goals.

Performance Tasks can be differentiated by content, process, and/or product to address various learner profiles. Students will have a choice for the Performance Task as to how they want to show their mathematical reasoning to their classmates. They might choose to break down the “race” into steps and show their classmates what they did at each stage of the race by using a storyboard or PowerPoint presentation, or they might choose to make a large poster that shows the big picture and the route that they chose from beginning to end. Students should be given high- and low-tech options to compose in multiple media, such as text, speech, drawing, illustration, comics, or storyboards. Students will use websites to aid them in comprehending mathematical concepts and will use drawings and equations to solve problems. Students can use graphic organizers, such as the Eyes and Ears protocol, concept mapping with *Inspiration*, or drawings by hand, checklists, sticky notes, and mnemonic strategies such as the SOHCAHTOA mnemonic device (see p. 5.8.20) to better understand and demonstrate comprehension of the material. Opportunities for collaboration and whole-class discussion should be provided as needed.
Literacy and Numeracy Across Content Areas

Reading
Students will read the instructions carefully in order to complete all assessments. Students will also read and interpret a variety of math texts, such as graphs and word problems, in order to solve problems or create solutions. Students will also read about the real-world applications to the mathematical concepts that they are studying.

Writing
Students will engage in writing activities through Do Now activities or Exit Tickets. They will also need to write to express their reasoning and to prove their answers. For the Performance Task, students will write reflections about their work. They might also choose to write out their mathematical thinking about how they solved the Performance Task.

Speaking and Listening
Students will speak with their teacher and classmates in order to complete all of the assignments in this unit. Students should share their reasoning with their classmates and build on the ideas of their classmates to clarify their own thinking. For the Performance Task, students will create presentations for their classmates that explain the best solutions they found to “The Extraordinary Race” question. Students will listen to their classmates’ explanations so they can later reflect on whether or not they came up with the best solutions to their own problems.

Language
Students will use content-specific vocabulary to explain the mathematical concepts discussed in this unit.

Numeracy
Students will explore right triangles and angle and leg measurements throughout this unit. They will use their understanding of triangles and trigonometry to solve problems.
Resources (in order of appearance by type)

Print


Websites

Lesson 2

www.youtube.com/watch?v=w1NSXRMaB94

Lesson 3
www.youtube.com/watch?v=eJ6ky97LaBc

Lesson 5
“The Distance Formula.” *Cut the Knot*. Alexander Bogomolny. 2016.
www.cut-the-knot.org/pythagoras/DistanceFormula.shtml

Lesson 6
www.khanacademy.org/math/geometry/similarity/intro-to-triangle-similarity/v/similar-triangle-basics

Lesson 8
www.desmos.com/calculator

Lesson 10
www.youtube.com/watch?v=L5b2gtsGkcs

http://work.chron.com/jobs-use-clinometer-26847.html

Lesson 12
www.youtube.com/watch?v=4Q0LyGsoZ04

Mathway.com is an online resource for Algebra and Geometry. It is a live website where students can enter information (equations, radicals, problems, etc.) and get solutions.
Materials

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<tr>
<td>Lesson 6</td>
<td>Similar Right Triangles Facts</td>
<td>Google Drive</td>
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<tr>
<td>Lesson 6</td>
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<td>(DYS/SEIS educators only)</td>
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<tr>
<td>Lesson 7</td>
<td>Special Right Triangle Investigation 1</td>
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<tr>
<td>Lesson 7</td>
<td>Special Right Triangle Investigation 2</td>
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* The Extraordinary Race was inspired by the “Sapphire Heist” by Diane Stobbe and Renee Handfield. It was revised by Martin Fechner, Don Symes, Tim Gartke and Jeremy Klassen, then adapted for this Guide. Permission for use of this Performance Task was granted by the originator, the Edmonton Public Schools, District 7, of Alberta, Canada.
PREREQUISITES: Math skills needed for this unit

*Right Triangles and Trigonometry Ratios* is the fourth unit in the Winter Season of Geometry. The prerequisite math skills summarized below are taught in units that precede this unit (see Scope and Sequence chart). The following skills will be needed for students to successfully complete this unit.

Students should know:

- How to solve algebraic expressions with radicals
- How to factor and find multiples of numbers
- How to identify congruent triangles and their postulates, including:
  - Side-Angle-Side (SAS), Side-Side-Side (SSS), Angle-Side-Angle (ASA),
  - Angle-Angle-Side (AAS), and Hypotenuse-Leg (HL)

Outline of Lessons

Introductory, Instructional, and Culminating tasks and activities to support achievement of learning objectives

INTRODUCTORY LESSON

*Stimulate interest, assess prior knowledge, connect to new information*

Note: Teachers should vet all videos included in this unit according to program standards and create templates or graphic organizers for students to monitor their comprehension of the material.

Lesson 1

Introduction to the Pythagorean Theorem

**Goal**

Students will solve equations and discover the Pythagorean theorem.

**Do Now** (time: 5 minutes)

The teacher will give students the “Right Triangles Pre-Assessment,” Activity Sheet on p 5.10.1 of the Supplement. Students will look at the sheet and make notes about what they notice. The teacher will ask students:

Why do you think right triangles are important?

What do you notice about the angles inside the triangles?

What do you notice about the lengths of the sides?

Students will put their names on the top of their papers before trading their notes with partners, and
they will add to the partner’s list. Students will continue trading papers with other students, adding to each other’s lists until students feel like they have nothing else to add. When the teacher thinks that students have created long enough lists, papers will be returned to their owners and students will review their classmates’ notes.

**Hook** (time: 5 minutes)
The teacher will lead a discussion about right triangles and why they matter. For example, the teacher should mention that in construction, this is how workers know that their structure is safe. Bridges, buildings, and other structures need right triangles in order to be structurally sound. Ask students to think of other examples where a triangle is used for stability. (They might think of a tripod for a camera.) The teacher will show pictures of triangles to explain to students that triangles are stable structures. The illustration of the bridge below, which is stabilized by many triangles, could be an example that the teacher shows.

![Image of a bridge with red dots indicating triangles](image)

The red dots show one of the many triangles in the bridge.

**Presentation** (time: 5 minutes)
The teacher will put students in pairs and hand each pair of students a task card. Task cards are located on p. 5.10.2 of the Supplement. The teacher will explain to students that they are going to attempt to solve these problems however they can. The teacher will give students tools such as graph paper, rulers, and protractors.

**Practice and Application** (time: 20 minutes)
Without being directly told what the Pythagorean theorem is, students will try to figure out their tasks by using it.

The teacher will walk around the room to guide students in their thinking. The teacher should allow students to struggle with the problem for a little while and coach the students not to give up. If students are really struggling, the teacher might tell students that they could try drawing out their task in an attempt to solve it. The teacher will remind students of the materials that are available for them to use. The teacher will monitor discussion so that he or she knows what points to bring up during the review and assessment part of the lesson. This activity might take some time because students are not aware of the mathematical equation of the Pythagorean theorem that can be used to quickly solve these problems.

**Review and Assessment** (time: 20 minutes)
Students will share their tasks and what they learned, even if they did not come to a correct answer. They will explain how they approached the task and what they did to try to solve it.
After students have shared their findings, the teacher will explain the Pythagorean theorem to students. The teacher must use the correct vocabulary when discussing right triangles (Hypotenuse, Leg A, Leg B), so that students have the content-specific vocabulary to continue throughout the unit. While the teacher is using this correct vocabulary, he or she will begin posting a word wall with these terms. As the teacher posts the word wall, students will take notes in a T-chart. On one side of the T-chart, students will put the vocabulary word. On the other side of the T-chart, students will create a drawing that relates to the vocabulary word.

When students are done filling out their T-charts, they will correct any mistakes that they made while trying to solve their task cards. The teacher will ask students to reflect on why having a formula like the Pythagorean theorem is important for mathematicians to use. They should write down their answer before leaving class. The Pythagorean theorem makes solving side lengths of a triangle much easier.

Note: Students will not always have access to rulers, protractors, or other tools to solve a problem, or problems may not be drawn to scale, so they will need the equation to help them solve it.

INSTRUCTIONAL LESSONS

Build upon background knowledge, make meaning of content, incorporate ongoing Formative Assessments

Lesson 2

Practice Using the Pythagorean Theorem and Simplifying Radical Expressions

Goal
Students will apply their knowledge of the Pythagorean theorem to solve problems using it and will practice/review simplifying radical expressions.

Do Now (time: 5 minutes)
The teacher will give students multiple task cards (see the “Task Cards” Activity Sheet on p. 5.10.2) from the previous lesson and select one to use as a review of the Pythagorean theorem. The teacher should use a visual, labels, and essential vocabulary to support student understanding. The teacher will tell students to use the Pythagorean theorem to solve the rest of the problems. Students will notice that they can solve the problems much faster than they could the day before.

Hook (time: 10 minutes)
The teacher will have students brainstorm orally, in writing, or through drawing the people who may use the Pythagorean theorem or situations that require it. Then, the teacher will show students how people use the Pythagorean theorem in their daily lives. The teacher can develop his or her own examples or use the following examples: how GPS uses the Pythagorean theorem or how video game designers use the Pythagorean theorem. The teacher can do a read-aloud, jigsaw reading, or close reading of the following article, which makes note of uses of the Pythagorean theorem.

See: GPS, Construction and Video Games—Everyday uses for Pythagoras’s theorem
www.theguardian.com/science/shortcuts/2016/feb/10/
gps-construction-and-video-games-everyday-uses-for-pythagorass-theorem
The teacher could also show students a video that shows how video game designers use the Pythagorean theorem to design video games and have students identify how the theorem is used.

**SEE:** Example 1—Pythagorean theorem in video game design
www.youtube.com/watch?v=w1NSXRMaB94

**Presentation** (time: 10 minutes)
The teacher will explain that students are going to practice using the theorem today and that they will also practice solving radicals because $C^2$ is not always a perfect square. For example, it is difficult to solve for $C$ when $C^2=75$ because the square root of 75 is not a whole number.

The teacher will review how to simplify radical expressions and post and review content-specific vocabulary, processes, and the underlying mathematical principles associated with radical expressions. The teacher will provide time for review of factoring skills so that students know how to solve these problems. The teacher will put examples of these problems on the board and show students how to solve them. The teacher should model the process for solving a radical expressions and have students document the steps they will follow in the practice and application section. The teacher is encouraged to document the steps for future student reference.

**Note:** If students need extensive review on factoring, the “Simplifying Radicals” Activity Sheet in the Supplement on p. 5.10.3 may be used by for students as practice.

**Practice and Application** (time: 25 minutes)
Students will be given word problems of real-life situations to solve using the Pythagorean theorem (see the “Pythagorean Theorem Word Problems” Activity Sheet on p. 5.10.4 of the Supplement). These word problems are situations that students could find themselves needing to solve in their lives. The teacher should walk around the classroom and identify students’ common points of error and discuss these errors as a class in order to correct them.

**Review and Assessment** (time: 5 minutes)
Each student will be assigned one word problem to share with the class. Students could go to the board and use drawings to illustrate to their classmates how they solved the problem, or they could create a poster that they can post in the classroom that explains how to solve the problem. These posters could remain posted in the classroom to help students remember how to use the Pythagorean theorem to solve problems. Each student needs to explain his or her reasoning and how he or she solved the problem. The teacher should make corrections to student thinking and explanations as needed.

**Extension**
The teacher might decide that students need an entire day of review about how to solve radicals. If students are struggling during the presentation portion of this lesson, the teacher may decide to spend a day reviewing how to solve radicals and move the remainder of the lesson to the following day. The Activity Sheet “More Practice with Radicals” is located on Google Drive in the Geometry Chapter resources and contains a number of worksheets.

**SEE:** More Practice with Radicals
Lesson 3

Proof of the Pythagorean Theorem

Goal
Students will apply their knowledge of the Pythagorean theorem to prove that it is true for all right triangles.

Do Now (time: 5 minutes)
The teacher will begin by reviewing the theorem and terminology needed to complete the Do Now. The teacher will put three right triangles on the board with side lengths \(a\) and \(b\) labeled for each. The teacher will ask students to solve for the hypotenuse for each triangle. Students should use the Pythagorean theorem to do this, and they will likely be able to do this quickly since they accept the formula of the Pythagorean theorem to be true. Today’s lesson will ask them to prove that it is true.

Hook (time: 5 minutes)
The teacher will explain to students that long before Pythagoras proved his theorem, ancient Egyptians knew that they could build a perfect right triangle with the measurements of 3, 4, and 5. The teacher can show the 3 minute video: “Pythagoras Theorem and the Ancient World” to students that explains how Egyptians used knotted ropes to build pyramids. Before watching the video, students should take a white lined piece of paper and fold it in half. On one side of the paper they will write Egyptians and on the other side they will write Babylonians. While watching the video, they will take notes on what each group noticed about triangles before Pythagoras wrote his theorem. After watching the video, the teacher will ask students to explain what they learned from the video.

SEE: Pythagoras Theorem and the Ancient World
www.youtube.com/watch?v=eJ6ky97LaBc

Presentation (time: 10 minutes)
The teacher will explain to students that they are going to prove the Pythagorean theorem using the area of squares and will tell students that proving theorems can increase understanding. The lab that students are about to do will walk them through the process of proving the Pythagorean theorem.

Note: This lab requires students to cut out shapes. The teacher might want to have the shapes already cut out if students aren’t allowed to use scissors.

Practice and Application (time: 30 minutes)
The teacher will give students the “Proof of the Pythagorean Theorem Lab” Activity Sheet on pp. 5.10.5-5.10.10 in the Supplement. Students will work with partners to complete the lab. The lab will guide students through proving that the Pythagorean theorem applies to all right triangles. While approaching each task in the lab, the students will discuss their thoughts with partners and complete the task at hand. The teacher will circulate around the room to ensure that students are focused and to answer questions that students have. While working on the lab, students should realize that the Pythagorean theorem applies to all right triangles and can be used to find the side lengths of all right triangles.

Review and Assessment (time: 5 minutes)
Students will explain in writing or through illustration how finding the area of squares in this lab allowed
them to prove the Pythagorean theorem. Students should turn their writing or visual explanation into their teacher at the end of class so that the teacher can assess the students’ understanding.

Lesson 4
The Converse of the Pythagorean Theorem

Goal
Students will apply their understanding of the converse of the Pythagorean theorem to prove that triangles are not right triangles.

Do Now (time: 5 minutes)
The teacher will draw a triangle on the board with leg lengths of 6 inches and 7 inches. The students should solve the hypotenuse in the form of a radical to practice the skill taught in the previous class. Once students have solved the problem, the teacher will ask students:

Why is solving this problem more challenging than solving for the hypotenuse of, say, a 3, 4, 5 triangle?

Hook (time: 5 minutes)
The teacher will draw another triangle on the board with lengths labeled 6, 7, and 8 inches. The teacher will ask students to turn to partners and discuss whether or not this is a right triangle and how they know. The teacher will give students graph paper to draw this triangle so that they can see what the triangle looks like. They will also be given protractors and rulers to double check their work. They know that a right triangle must have one angle that is 90°. Students will explain their reasoning to the class after talking to their partners about it. The teacher will pose the question:

How is this question about triangles different from questions we have looked at in the past two lessons?

Students should notice that while in previous lessons they had to determine the length of an unknown side of a right triangle, now we know all three sides and must determine whether or not the triangle is a right triangle. The teacher will tell students that sometimes triangles are not drawn to scale and might appear to be right triangles when in reality they are not. We must rely on using the Pythagorean theorem to find out whether or not the triangle is, in fact, a right triangle.

Note: The teacher will use content-specific vocabulary and will refer to the word wall when reminding students about what they already know about right triangles and discussing how this problem is different.

Presentation (time: 5 minutes)
After students explain that the triangle in the hook is not a right triangle, the teacher will explain that we can figure that out by thinking about the converse of the Pythagorean theorem. If \( A^2 + B^2 = C^2 \) is a right triangle, then when \( A^2 + B^2 \) does not equal \( C^2 \), the triangle is NOT a right triangle. The teacher will demonstrate other examples of this on the board for students. The teacher will call on students to determine whether or not the triangles he or she draws on the board are right triangles and will ask students to explain why they are or aren’t. Since the purpose of this activity is for students to understand the concept of the converse of the Pythagorean theorem, the teacher will use small numbers as the lengths of the sides of the triangles so that students can easily do the math in their heads or on paper.
Practice and Application (time: 30 minutes)
Students will practice the converse and the Pythagorean theorem by working on the “Converse of the Pythagorean theorem” Activity Sheet on pages 5.10.11 and 5.10.12 of the Supplement. The teacher should circulate around the room to make sure that students are using correct mathematical reasoning to solve the problems. Students will be given options for participation. They may work alone, with partners, or with a small group.

Review and Assessment (time: 10 minutes)
Students will create their own right triangles and non-right triangles and use the Pythagorean theorem and the converse of the Pythagorean theorem to prove that one triangle is a right triangle and the other triangle isn’t. Students can use tools such as a computer, graph paper, or protractor to draw their triangles. After drawing their triangles, students will trade their work with partners. The partner will determine which triangle is a right triangle and which triangle is not a right triangle by using the Pythagorean theorem and the converse of the Pythagorean theorem. Students will compare their answers with the student who originally created the triangles in order to check their work and reasoning.

Lesson 5
Distance Formula

Goal
Students will calculate the distance between two points using the Pythagorean theorem.

Do Now (time: 5 minutes)
The teacher will draw or project a picture of a map on the board like the one on top of the next page. The map will have two points on it. The teacher will ask students:

  How would you be able to find out the shortest distance between the two points?
  How do you know that this is the quickest route?

The students should say that the Pythagorean theorem would help them find the distance between the two points. The diagonal is the shortest distance. If you added the squares of the two legs, it would be a greater distance than the square root of the hypotenuse.

Hook (time: 5 minutes)
The teacher will lead a discussion or have students do a Think/Pair/Share about why the distance formula matters in real life. The teacher will ask students:

  What are some reasons why people should care about how to calculate the distance between two points?
  In what situations would we want to know which route to take to get to a destination?
  What kinds of jobs might require people to get between two points in the shortest amount of time?
Presentation (time: 15 minutes)
The teacher will show students how to use the distance formula when students are given two points.

The formula to find the distance is:
\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

The teacher will model how to put any given points into the distance formula (above) in order to solve the distance between the two points. The teacher will show students how to put negative points into the formula and review rules for subtracting negative numbers.

Practice and Application (time: 20 minutes)
Students will solve a problem on the following website that asks them to use the distance formula to solve a problem:

**SEE:** The Distance Formula
www.cut-the-knot.org/pythagoras/DistanceFormula.shtml

The teacher will have each student work on the same problem or divide students into pairs and ask them to work on different problems and solutions. The teacher will circulate around the room and help students who are having trouble. The puzzles have varying degrees of difficulty; the teacher can differentiate in the classroom based on student need. When students have solved the problem, they can click on the solution box to check their answers. If they did not get the correct answer, the teacher will instruct students to go back to the puzzle to see where they went wrong.

The teacher will inform students that they will present to the class in the Review and Assessment activity.

Review and Assessment (time: 10 minutes)
Students will share out their answers to the games that they played using the distance formula and be able to explain verbally or in writing how they were able to solve the problem, or the teacher will ask them to explain where they made an error if they found that they did not find the correct solution. If they had trouble finding a solution, they should explain how they were able to change their thinking in order to come to the correct answer. They should explain what they would do differently if they were to encounter this same problem again.
Lesson 6

Similar Right Triangles

Goal
Students will prove the similarity of right triangles.

Do Now (time: 5 minutes)
The teacher will give the lengths $A = 5$, $B = 6$, and $C = 8$ of yarn to students. They will be asked to prove if the triangle is a right triangle or not. (It is not because $25 + 36$ does not equal $64$). Students will share results and their problem solving strategies.

Hook (time: 5 minutes)
The teacher will show students the graphic on the top of the “Similar Right Triangle Activity 1” on p. 5.10.13 in the Supplement. The teacher will then ask students what they notice about the angles of the triangles and the side lengths of the triangles and record the students’ ideas on the board.

Presentation (time: 15 minutes)
The teacher will show students a Khan Academy video that explains similar right triangles.

See: Intro to Triangle Similarity
www.khanacademy.org/math/geometry/similarity/intro-to-triangle-similarity/v/similar-triangle basics

Students will use the “Eyes and Ears Protocol” Activity Sheet to take notes (p.5.10.14 of the Supplement). In pairs, one student will write down what he or she sees, while the other will take notes on what he or she hears. Students will then take turns sharing their notes from the video as well as their thoughts with each other.

Practice and Application (time: 20 minutes)
Students will complete the “Similar Right Triangles Activity 2” in the Supplement on p. 5.10.15.

The teacher will go over the “Similar Right Triangle Facts” handout (available on Google Drive) with students or have the facts posted in the classroom. The teacher and students may also wish to use the resource on similar right triangle side lengths available on the Math Warehouse website.

See: Similar Right Triangle Facts
Math Warehouse
www.mathwarehouse.com/geometry/similar/triangles/geometric-mean.php

The teacher will circulate around the classroom to answer any questions that students have.

Review and Assessment (time: 10 minutes)
The teacher will review the handout with students and answer any questions that arose. Students will complete an Exit Ticket that answers the following question:

What does it mean for right triangles to be similar?

Students should respond that similar right triangles have congruent corresponding angles and congruent sides.
Lesson 7

Special Right Triangles

Goal
Students will discover rules about special right triangles by analyzing similarities among triangles.

Do Now (time: 10 minutes)
The teacher will give students the “Special Right Triangle Investigation 1” Activity Sheet in the Supplement on p. 5.10.16 and have students work in pairs to complete it. Students should write down what they notice, what they have questions about, and/or what they know to be true about the relationships of the sides of these triangles.

Hook (time: 10 minutes)
Students will verbalize to the class and the teacher their reflections from the Do Now. The teacher will encourage students to share their rules (#4 from handout) with the class. The teacher will tell students that the type of triangle they are looking at is called an isosceles right triangle. The teacher will ask students to make sure their rule works in all isosceles right triangles.

*In isosceles right triangles, if the legs have length L, then the hypotenuse has length $L\sqrt{2}$.*

The teacher will ask students to go backwards to make sure their rule works in the examples on their papers. Teachers should reference the steps students took in Lesson 3 where they proved the Pythagorean theorem as a guide to how to prove their rule from the handout.

Presentation (time: 5 minutes)
The teacher will add the term *isosceles right triangles* to the word wall in the classroom with a visual for students. The teacher will ask students to repeat the rule for finding the hypotenuse of this type of triangle. The teacher will then tell students that they’re going to do another investigation to determine the rules in another special right triangle. The goal will be to come up with a rule for this type of triangle, just as they did in the Do Now for the isosceles triangle. The teacher will review the steps that students will take, as outlined in the “Special Right Triangle Investigation 2” Activity Sheet located in the Supplement on p. 5.10.17.

Practice and Application (time: 15 minutes)
Student will work with partners to complete the second investigation and come up with a rule or rules for 30-60-90 triangles:
- Short Leg = $\frac{1}{2}$ hypotenuse
- Long Leg = $\frac{1}{2}$ hypotenuse $\cdot \sqrt{3}$
- Long Leg = Short Leg $\cdot \sqrt{3}$

Students should share and discuss their rules in the following activity.

Review and Assessment (time: 10 minutes)
The teacher will review with students the rules they came up with to express the relationships in 30-60-90 triangles. As an assessment, the teacher will draw a 30-60-90 triangle on the board with the long leg labeled (17) and ask the students to work with partners to use their rules to quickly find the short leg and the hypotenuse.
The teacher should have students reflect on how they worked with their partners and the steps they took to effectively collaborate. The teacher will also ask a student to add this type of triangle and the rule that was discovered to the word wall.

Lesson 8

Introducing Trigonometry

Goal
Students will discover that the ratios of the legs and hypotenuse of similar right triangles and their relationships with the angles form the basis for the study of trigonometry.

Do Now (time: 5 minutes)
The teacher will give students the ordered pairs (-5, 3) and (5, 1). The students will need to determine the distance between these points and round to the nearest tenth (√116≈10.8).

Hook (time: 10 minutes)
Students will complete the “Discovering Trigonometry” Activity Sheet in the Supplement on pp. 5.10.18-5.10.19. The teacher will provide graph paper, rulers, and protractors for students. The teacher will model how to complete the task and ensure understanding of vocabulary terms and concepts needed to complete the problems. Students will share their answers with the class and review the correct answers with the teacher. If students have a misunderstanding, the teacher will ask students to explain what they did to find the answer to help students understand where they made a mistake. The teacher should encourage students to explain their correct reasoning to students who had a misunderstanding so that students are learning from each other and aren’t relying on the teacher to give them the correct answer. The teacher can have students repeat the correct steps that they should have taken to reinforce understanding.

Presentation (time: 25 minutes)
The teacher will tell students that we sometimes need to find the length of a side of a right triangle when we do not know the lengths of two of the sides, meaning that we can’t use the Pythagorean theorem to solve for the side. In the Do Now, we were able to figure out the distance between two points by using the Pythagorean theorem, but that is only because we know the lengths of the other two sides of the triangle. The teacher will tell students that we can use trigonometry to discover the lengths of the triangle’s sides if we know an angle measurement other than the 90° angle.

The teacher will explain that the ratios of the sides of a triangle have special names. Before explaining the ratio names, the teacher will draw a right triangle on the board and label all the sides and all the angles.

The teacher will introduce the labels of opposite and adjacent since students are already familiar with hypotenuse. Select an angle and identify the side opposite that angle. Using the same angle, identify a side that is adjacent to that angle.

The \( \sin (\text{sin}), \cos (\text{cos}), \) and \( \tan (\text{tan}) \) functions show how the sides and angles are related to each other. Introduce the names of the ratios to students one at a time.
As the teacher is introducing this new vocabulary, s/he should add the words to the word wall. Students will take notes on a T-chart, writing vocabulary words on one side and drawing a picture to represent them on the other.

Walk students through how to find the sine, cosine, and tangent of an angle in a right triangle by using the ratios. Students should continue taking notes and drawing pictures on their T-charts to help them remember these terms.

The teacher will introduce students to the SOHCAHTOA mnemonic device:

- Sine = Opposite / Hypotenuse
- Cosine = Adjacent / Hypotenuse
- Tangent = Opposite / Adjacent

This mnemonic device and its meaning should also be added to the word wall and to students’ T-charts.

The teacher will then complete a couple of examples on the board with each of the trigonometry functions. The teacher can show students how to do this on an online graphing calculator such as Desmos.

SEE: Desmos Online Graphing Calculator
https://www.desmos.com/calculator

Practice and Application (time: 10 minutes)

The teacher will ask students to go back to the triangles they created during the discovery activity and find the sine, cosine, and tangent for every angle in their triangles.

Review and Assessment (time: 5 minutes)

Each student will share an angle measure from one of his/her triangles, and give the sine, cosine, and tangent for the angle.

Lesson 9

Trigonometric Ratios

Goal

Students will continue to discover that trigonometric ratios come from similar right triangles and that they are constant. Students will use these ratios to find missing legs and angles in right triangles.

Do Now (time: 5 minutes)

The teacher will ask students to write the definition or anything that they remember from yesterday’s lesson about sine, cosine, and tangent. Students should think for a few minutes and then turn to their partners to compare notes.

Hook (time: 5 minutes)

The teacher will ask students if they can find the measure of a missing leg in a triangle using what they’ve learned about trigonometry functions.
For example, the teacher draws the top triangle (at left) on the board and asks students if they can find the value of $x$.

**Presentation** (time: 5 minutes)
The teacher will also tell students that they can find the measure of a missing angle using their knowledge of trigonometry ratios. Students can also use their calculators for these. Use the bottom triangle (at left) to help students make the connection.

**Practice and Application** (time: 20 minutes)
Students will complete the “Right Triangle Ratios Lab” on pp. 5.10.20-5.10.21 in the Supplement. They will discover that trigonometric ratios are constant based on the concept of similarity. The teacher should walk around the classroom to listen to students’ conversations about the lab. The teacher should try to allow students to make the discovery on their own, but should guide thinking if students need help.

**Review and Assessment** (time: 20 minutes)
The teacher will lead a discussion about the lesson and ask students to share their conclusions with the class. The teacher should clear up any misconceptions that students have. Once the teacher clears up misconceptions, students will do a self-assessment. They will either reflect on how they discovered the correct conclusion and list the steps that they took, or they will reflect on where they went wrong in their thinking and how they would approach the problem differently if they were given it again.

**Lesson 10**

**Clinometers and Ceiling Height (Part 1)**

**Goal**
Students will build clinometers in order to estimate height.

**Do Now** (time: 5 minutes)
The teacher will post the following question on the board for students to consider:

How can we measure the height of the ceiling in this classroom without a ladder or anything else to stand on?

Students will complete a think-pair-share activity.

**Hook** (time: 10 minutes)
The teacher will show students a video that explains how we can find the height of tall objects. The video below explains what clinometers are and how to use them to calculate the height of tall objects.

If the teacher wants students to take notes while watching the video, students can write down the steps...
that they must take to build a clinometer. Students will be given instructions later in the lesson, so if the teacher decides not to have students take notes, students can simply watch the video to see a visual of what the finished product will look like.

After watching the video, the teacher will tell students that they will create their own clinometers so that they can use trigonometry to find the heights of objects.

**SEE:** How to Make and Use a Clinometer
www.youtube.com/watch?v=L5b2gtsGkcs

**Presentation** (time: 15 minutes)
The teacher will model for students how to build a clinometer, and walk students through the instructions step by step as he or she builds it.

**Practice and Application** (time: 15 minutes)
The teacher will provide students with the diagram for building a clinometer (Step 1 of the “Ceiling Height Estimation Lab” Activity Sheet, located on p. 5.10.22 of the Supplement). The students will build their own with partners or individually, depending on the availability of resources.

**Note:** Students will only build the clinometer today (STEP 1). They will do the rest of the lab in Lesson 11.

**Review and Assessment** (time: 10 minutes)
Once students have built their clinometers, the teacher will ask:

Why might people need to use these? In what situations would we need to measure the height of something that we can't otherwise measure?

Examples of careers that use clinometers can be found at:

**SEE:** What Jobs Use a Clinometer?
http://work.chron.com/jobs-use-clinometer-26847.html

The teacher will ask students:

If you could create an even better clinometer, how would you do that?

The teacher will then ask students to sketch what a “superior” clinometer would look like and to explain how they would build it.

**Extension**
Students will continue to work with their clinometers the following day.

**Lesson 11**

**Clinometers and Ceiling Height (Part 2)**

**Goal**
Students will use their clinometers and knowledge of trigonometry to measure the height of objects.
**Do Now** (time: 5 minutes)
The teacher will ask students to estimate how high they think the ceiling of the room is. What reasoning are they using to estimate that height? How tall do they think the building is? Why do they think that?

**Hook** (time: 5 minutes)
The teacher will brainstorm a list of possible things that the students will be able to measure with the clinometers that they made the previous day. Students will estimate the heights of the objects that they want to measure so that they can see how accurate their estimations were once they are able to measure the objects.

**Presentation** (time: 5 minutes)
The teacher will demonstrate for students how to use the clinometer to measure the height of an object and will give students the clinometers that they created the previous day.

**Practice and Application** (time: 30 minutes)
Students will continue the “Ceiling Height Estimation Lab” on pp. 5.10.22-5.10.23 using the clinometers that they created yesterday. The teacher will inform students that they will present their results to the class in the following activity.

**Note:** After students are done estimating the ceiling height, they should practice their math skills by measuring the height of other objects in the room, or if possible, outside. Students might measure the height of a tree or the height of the building.

**Review and Assessment** (time: 10 minutes)
Students will share their estimations with the class and the mathematical reasoning that they used to measure the height of objects in the room or outside. The teacher will correct any problems with students’ thinking after they are done explaining how they arrived at an answer. If students came up with different answers for the ceiling height, they should discuss why this might have happened.

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**CULMINATING LESSONS**

*Includes the Performance Task, i.e., Summative Assessment—measuring the achievement of learning objectives*

**Lesson 12**

**Extraordinary Race (2 days)**

**Goal**
Students will apply their understanding of right triangles and their properties to find the fastest route through an obstacle course.

**Do Now** (time: 5 minutes)
The teacher will ask students:

How can you determine the shortest distance between points? What mathematical skills could you use?

Students will discuss their answers with partners before sharing them with the class.
**Hook** (time: 5 minutes)
The teacher will show students a clip from the television show *The Amazing Race* so students understand the concept of the activity they are being asked to complete. The first minute of the following video explains the premise of the show:

SEE: *The Amazing Race Intro Season 1-24*  
www.youtube.com/watch?v=4Q0LyGsoZ04

Students must “race” their classmates through the course and beat them to the finish.

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**Presentation** (time: 30 minutes)
The teacher will explain the Performance Task to students and will provide students with a map of the course. The teacher will explain to students that there are two parts to this assessment, Part A (map shown above and also in project packet) and Part B (in project packet.) Students must complete Part A before they complete Part B. The teacher will explain to students that they will present the route they chose and explain their reasoning behind choosing the path that they did. The “Extraordinary Race” Project Packet is located in the Supplement on pp. 5.10.24-5.10.33. The ANSWER KEY is on pp. 5.10.34-5.10.38. Permission for use of these materials was granted from the Edmonton Public Schools District No 7 in Alberta, Canada.

Once the teacher completely explains the assessment to students, the teacher will create a rubric with the students to assess their work. The teacher and students should also brainstorm ways for the students to present their projects to their classmates and to explain their reasoning. Posters, PowerPoint presentations, storyboards, or Prezis are some options that could be given to students.
Practice and Application (time: 65 minutes—15 minutes Day 1, 50 minutes Day 2)
With partners, students should work through their options for completing the course in the quickest amount of time. The teacher should walk around the room to support students as they complete the Performance Task. They will need to apply their knowledge of the Pythagorean theorem and trigonometric ratios.

Review and Assessment (time: 5 minutes)
At the end of each class period that students are working on this project, the teacher should ask students to write down one question that they have and one thing that they were successful doing.

Lesson 13
Extraordinary Race—Presentations (2 days)

Goal
Students will create presentations and explain their mathematical reasoning in presentations to the class.

Do Now (time: 5 minutes)
Students will talk to their partners and decide how they want to explain the route they chose to the class and the mathematical reasoning behind it. Students might want to break the course into a series of steps in a PowerPoint, or they might create a storyboard that shows the steps that they would take on each stage of the journey.

Hook (time: 5 minutes)
The teacher will lead the class in a discussion about what makes a good presentation. The teacher will make a list of these characteristics on the board for students to keep in mind when presenting.

Presentation (time: 10 minutes)
The teacher will review the rubric that students created in the previous lesson. The teacher will answer any questions that were written on the Exit Tickets the previous day.

Practice and Application (time: 80 minutes—35 minutes Day 1, 45 minutes Day 2)
Students will create their presentations with their partners. They will use the rubrics that they created as a guide to be sure that they are putting everything that they need to into their presentations. The teacher will guide students who are struggling and remind them to look at the rubric.

Students will practice their presentations before they present and should use the rubric to self-assess when they think that they are done. They should be given time to revise their work as needed. Students will present their projects to their classmates.

Review and Assessment (time: 10 minutes)
Students will write a self-reflection after all students have presented. They should answer the questions:

What did I learn from watching my classmates present their projects?
Do I still think that my solution was the best? Why or why not?
POST–UNIT REFLECTION

On meeting the Learning and Language objectives

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Connections to Empower Your Future
UNIT: Right Triangles and Trigonometry Ratios

Future Ready Connections
Throughout the unit, students are encouraged to experiment, reflect, and try new approaches to solving problems using equations and tools. This is especially apparent in Lesson 1 and Lesson 7.

Giving students the freedom to experiment will allow teachers to evaluate students on their initiative, self-direction, and accountability for completing tasks, actively engaging in critical thinking, and taking responsibility for their own discoveries. Teachers should reflect on whether or not youth stay on task without prompting and if they push themselves to thoroughly complete each activity, answer their own questions, and create a detailed final product instead of only addressing the minimum required information or waiting for explanations from the teacher.

Youth have many opportunities to strengthen their communication and listening skills through group discussions, oral reflections, written responses, and during partner work on their presentations. Both oral and written communication and explanations can be evaluated for clarity and effectiveness. Group discussions and partner work are also appropriate times to assess students’ ability to work effectively and respectfully with diverse teams by sharing responsibility for collaborative work and validating individual contributions made by team members.

Teachers are encouraged to use the Future Ready Rubric to evaluate students’ growth and are encouraged to have students self-evaluate their progress using the Future Ready Rubric.

PYD/CRP Connections
This unit reflects many aspects of Culturally Responsive Practice and Positive Youth Development by utilizing familiar and interesting examples of triangles and by asking students to activate their prior knowledge and experience. For example, in Lesson 1’s Hook, students are asked to think of examples where a triangle is used for stability. The unit utilizes real-world examples for studying triangles and trigonometry such as when designing bridges or video games and when using GPS. This helps youth create meaningful connections to the topic and deepen their understanding of how this knowledge appears in the world. Youth are also encouraged to work both independently and collaboratively as they experiment with equations and proofs. These opportunities allow students to be critical thinkers and to depend on their experiences and on each other to support their learning and the learning of their peers.

The unit encourages exploration, experimentation, and reflection by scaffolding activities so that students are active participants in their learning and are a part of the discovery process. By encouraging youth to take risks and to be part of the discovery process, teachers are grounding the tasks in the students’ strengths and allowing students to have a voice in the classroom and in their own learning. Teachers are encouraged to differentiate lessons and engage youth as resources in the classroom. In Lesson 12, students will help create the rubric that will be used to assess their Performance Task. Including youth in the evaluation and assessment process encourages youth voice, self-reflection, and engages youth.

Career Exploration Connections
This unit includes several connections to career development awareness such as in Lesson 2’s Hook, Lesson 5’s Hook, and Lesson 10’s Review and Assessment. These activities increase students’ awareness of the career
“By encouraging youth to take risks and to be part of the discovery process, teachers are grounding the tasks in the students’ strengths and allowing students to have a voice in the classroom and in their own learning.”

fields that need to understand and utilize the Pythagorean theorem and trigonometry. Teachers should consider expanding these activities and allow students to explore these career fields in more detail. Students can use the MassCIS (Massachusetts Career Information System) to better understand the types of careers that use these mathematical concepts.

SEE: MassCIS (homepage)
https://portal.masscis.intocareers.org

In the search box, students can type in keywords such as “math.” This brings up the global search results page and one of the links will be “Occupations.” Click on this link to see the careers that utilize or require math skills. After clicking on a specific career, clicking on “Skills and Abilities” will show a detailed explanation of the skills and abilities required by this career field.

Teachers can also expand the career connections opportunities by working with students to identify types of jobs that use these mathematical skills on sites like Indeed and Career Builder. Use keywords such as “trigonometry,” “mathematical,” “mathematics,” and “measuring instruments” to search for jobs. Sample results for using the keywords “measuring instruments” might include an entry level job with a company that manufactures precision sheet metal and machined turbine engine components and a machine straightener who will utilize precision measuring instruments such as micrometers, dial indicators, and roundness gauges to check accuracy of work. Teachers and students can discuss the job postings and analyze what other skills, training, education, and experiences a person would need to have to enter this field.

Workplace Readiness Connections

Throughout this unit students are responsible for working collaboratively with their peers which provides opportunities for students to develop workplace readiness skills such as effective communication, cooperation, respect for others, teamwork, and time management. These skills are specifically required in Lesson 13 where students must work with partners to decide how they want to create and present their projects. Teachers are encouraged to specifically name the workplace readiness skills students will use while completing their projects and ask students to reflect on how they demonstrated those skills, what they could do to further improve them, and how they are transferable to other situations and experiences. Teachers are encouraged to use the Future Ready rubric to evaluate students’ growth and are encouraged to have students self-evaluate their progress using the Future Ready Rubric.

For Technical Assistance with Empower Your Future connections and lessons, please request support by submitting a Coaching Request ticket using the Coaching Feature on TeachPoint.
Right Triangle Pre-Assessment

Lesson 1

DIRECTIONS: Look at the right triangles below, making note of anything that you notice. Consider the following questions: Why do you think right triangles are important? What do you notice about the angles inside the triangles? What do you notice about the lengths of the sides?
## Task Cards
Lessons 1 and 2

**DIRECTIONS:** For the Lesson 1 presentation, the teacher will cut out the task cards, hand them out to students and have them solve the word problems. The cards are used again for the Do Now in Lesson 2.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Maria walked 3km west and 4km south. How far is she from her starting point?</td>
</tr>
<tr>
<td>2.</td>
<td>Laptop screen sizes are determined by the length of the diagonal portion of the screen; rounded to the nearest whole number. A laptop has a 10-inch width and the height measure 8 inches. Calculate the screen size.</td>
</tr>
<tr>
<td>3.</td>
<td>David leaves the house to go to school. He walks 200 meters west and 125 meters north. How far is he from his starting point?</td>
</tr>
<tr>
<td>4.</td>
<td>Mrs. Thornton is teaching a 5th grade class. She is standing 15 meters in front of Hilary. Jane is sitting 8 meters to Hilary’s right. How far apart are Mrs. Thornton and Jane?</td>
</tr>
<tr>
<td>5.</td>
<td>A wire is stretched from the top of an 8-foot pole to a bracket 5 feet from the base of the pole. How long is the wire?</td>
</tr>
<tr>
<td>6.</td>
<td>A 20-foot ladder leading against a wall is used to reach a window that is 17 feet above the ground. How far from the wall is the bottom of the ladder?</td>
</tr>
<tr>
<td>7.</td>
<td>The glass for a picture window is 8 feet wide. The door it must pass through is 3 feet wide. How tall must the door be for the glass to pass through the door?</td>
</tr>
<tr>
<td>8.</td>
<td>A 10-foot ladder is leaning against a wall. If the foot of the ladder is 3 feet from the wall, how high up the wall does the top of the ladder reach?</td>
</tr>
<tr>
<td>9.</td>
<td>How far from the base of a house do you need to place a 15-foot ladder so that it exactly reaches the top of a 12-foot wall?</td>
</tr>
<tr>
<td>10.</td>
<td>A baseball diamond is a square that is 90 feet on each side. What is the distance from home to second base?</td>
</tr>
</tbody>
</table>
Simplifying Radicals
Lesson 2

1. You should know that:
   \[ \sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \quad \text{AND} \quad \sqrt{a} \cdot \sqrt{b} = \sqrt{ab} \]

2. You should also have a list of perfect squares (a number that is the product of two equal integers).
   These are the first twelve:
   
   \begin{align*}
   4 &= 2 \times 2 \\
   9 &= 3 \times 3 \\
   16 &= 4 \times 4 \\
   25 &= 5 \times 5 \\
   36 &= 6 \times 6 \\
   49 &= 7 \times 7 \\
   64 &= 8 \times 8 \\
   81 &= 9 \times 9 \\
   100 &= 10 \times 10 \\
   121 &= 11 \times 11 \\
   144 &= 12 \times 12
   \end{align*}

   Simplifying radicals with perfect squares is easy because the answer is a whole number.
   Ex:  \[ \sqrt{25} = 5; \quad \sqrt{121} = 11; \quad \sqrt{64} = 8, \text{ and so on…} \]

3. When simplifying radicals, remember #1 above and try to find multiples that are perfect squares to make it easier. For example, find \( \sqrt{50} \)
   
   \[ \sqrt{50} = \sqrt{(25 \cdot 2)} \] (The goal is to find any multiple of 50 that’s a perfect square.)
   
   Based on #1 above:  \[ \sqrt{(25 \cdot 2)} = \sqrt{25} \cdot \sqrt{2} \]
   
   25 is a perfect square, and \( \sqrt{25} = 5 \) and \( \sqrt{2} \) is already in its simplest form.
   
   So, \( \sqrt{50} = 5\sqrt{2} \)

Here’s another example: Find \( \sqrt{96} \)
   
   \[ \sqrt{96} = \sqrt{(16 \cdot 6)} \]
   
   \[ \sqrt{(16 \cdot 6)} = \sqrt{16} \cdot \sqrt{6} \]
   
   \( \sqrt{16} = 4 \) and \( \sqrt{6} \) is already in simplest form
   
   So, \( \sqrt{96} = 4\sqrt{6} \)
Pythagorean Theorem Word Problems
Lesson 2

DIRECTIONS: Students will complete the following word problems using their knowledge of the Pythagorean theorem and their ability to simplify radicals. Each answer must be a whole number or a simplified radical. Students should use a blank sheet of paper to show their work.

1. The slide at the playground has a height of 6 feet. The base of the slide measured on the ground is 8 feet. What is the length of the sliding board?

2. The bottom of a 13-foot straight ladder is set into the ground 5 feet away from a wall. When the top of the ladder is leaned against the wall, what is the distance above the ground it will reach?

3. In shop class, you make a table. The sides of the table measure 36” and 18”. If the diagonal of the table measures 43”, is the table “square”? (In construction, the term “square” just means the table has right angles at the corners.)

4. A baseball “diamond” is actually a square with sides of 90 feet. If a runner tries to steal second base, how far must the catcher, at home plate, throw to get the runner “out” at second? Given this information, explain why runners more often try to steal second base than third base.

5. During a football play, DeSean Jackson runs a straight route 40 yards up the sideline before turning around and catching a pass thrown by Michael Vick. On the opposing team, a defender who started 20 yards across the field from Jackson saw the play setup and ran a slant towards Jackson. What was the distance the defender had to run to get to the spot where Jackson caught the ball?

6. In construction, floor space must be planned for staircases. If the vertical distance between the first and second floors is 3.6 meters, and a contractor is using the standard step pattern of 28 cm wide for 18 cm high, then how many steps are needed to get from the first to the second floor and how much linear distance (i.e., “width” or “base”) will be needed for the staircase? What is the length of the railing that would be attached to these stairs?
Pythagoras' theorem states that:
In any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Algebraically, the theorem is often stated as $a^2 + b^2 = c^2$
where $a$ and $b$ represent the length of each leg and $c$ represents the length of the hypotenuse.

**DIRECTIONS:** Complete the activities and examples that follow.

1. For each fact below, use the checkboxes on the right to indicate whether you understand this fact or have a question.

<table>
<thead>
<tr>
<th>FACT</th>
<th>I understand this fact.</th>
<th>I have a question about this.</th>
</tr>
</thead>
<tbody>
<tr>
<td>In a right triangle, the legs form a right angle. The hypotenuse is opposite the right angle.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>The legs of a right triangle form a right angle. (This is shown with a small box where the right angle is.)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>The hypotenuse is opposite the right angle (across from it). Each leg is adjacent to the right angle (next to the angle, touching it).</td>
<td></td>
<td></td>
</tr>
<tr>
<td>When using the Pythagorean theorem, the variable $c$ is always used for a hypotenuse. The variables $a$ and $b$ can be used with either leg.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Using the word bank on the right, add labels in the ovals and rectangles for the triangle.

<table>
<thead>
<tr>
<th>Fill in the ovals using these variables</th>
<th>Fill in the rectangles using these words</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>Hypotenuse</td>
</tr>
<tr>
<td>$b$</td>
<td>Legs</td>
</tr>
<tr>
<td>$c$</td>
<td>Right Angle</td>
</tr>
</tbody>
</table>
Proof of the Pythagorean Theorem Lab

Lesson 3

3. Use the numbers 7 and 5 for the “leg” sides of the triangle and fill in the boxes as indicated.

```
Start by writing the equation for the Pythagorean theorem (see p. 5.10.5).

Substitute \(a = 7\) and \(b = 5\).

Square \(a\) and \(b\), then add the two terms.

\[7^2 + 5^2 = c^2\]
\[49 + 25 = c^2\]
\[74 = c^2\]

For the final step, find \(\sqrt{74}\). Use a calculator to find your answer and write it in the box below.

\[\sqrt{74} = \sqrt{c^2}\]
```

4. Complete the steps in the boxes to the right.

```
Start by writing the equation for the Pythagorean theorem (see p. 5.10.5).

Substitute \(a = 20\) and \(b = 5\).

Square \(a\) and \(b\), then add the two terms.

For the final step, find \(\sqrt{c^2}\). Use a calculator to find your answer and write it in the box.

\[\sqrt{c^2}\]
```
5. Find the length of the hypotenuse (c) for the following seven triangles.

5-a. (c) length: 

\[ \begin{array}{c}
7 \\
8 \\
\end{array} \]

5-b. (c) length: 

\[ \begin{array}{c}
5 \\
5 \\
\end{array} \]

5-c. (c) length: 

\[ \begin{array}{c}
5 \\
12.3 \\
10.4 \\
\end{array} \]

5-d. (c) length: 

\[ \begin{array}{c}
5 \\
10 \\
6 \\
\end{array} \]

5-e. (c) length: 

\[ \begin{array}{c}
5 \\
15.07 \\
7.521 \\
\end{array} \]

5-f. (c) length: 

\[ \begin{array}{c}
14.1 \\
10.54 \\
\end{array} \]

5-g. (c) length: 

\[ \begin{array}{c}
5 \\
7.589 \\
16.79 \\
\end{array} \]
Proof of the Pythagorean Theorem Lab
Lesson 3

For a right triangle with a hypotenuse of length c, we will examine a geometric way of proving that this equation is true for right triangles.

**DIRECTIONS:** Begin by creating and cutting out 5 identical right triangles. Use construction paper or cut out the triangles supplied at the bottom of this sheet.

6. Use one triangle to measure the lengths of sides a, b, and c. Record the lengths in this table:

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
</table>

7. Now, find the values of $a^2$, $b^2$, and $c^2$ and fill them into the table below:

<table>
<thead>
<tr>
<th>$a^2$</th>
<th>$b^2$</th>
<th>$c^2$</th>
</tr>
</thead>
</table>

Does the equation $a^2 + b^2 = c^2$ hold true? Do others in the class arrive at the same result?

Cut out each of the 5 triangles below to use for the activities in this lab:
8. Rearrange the remaining 4 triangle cutouts so that they form a large square with a small square on the inside, like the illustration at right.

Let’s state the area of the square $ABCD$ algebraically in two different ways:

8-a. **FIRST WAY**

What is the Length of $ABCD$?

How do you find the area of a square?

What’s the factored expression for the area of the square $ABCD$?

What is the simplified expression for the area of the square $ABCD$?
8-b. SECOND WAY

What is the area of the square on the inside of $ABCD$?

What is the area of one of the four triangles in $ABCD$?

What is the total area of all four of those triangles?

What is the total area of the square $ABCD$, written as the sum of the small square and the 4 triangles? Simplify your expression.

9. Let’s state the two expressions that represent the area of the square $ABCD$.

<table>
<thead>
<tr>
<th>FIRST WAY</th>
<th>SECOND WAY</th>
</tr>
</thead>
</table>

Since these both represent the same square, we can say that both expressions are equal. Write an equation that states both of these expressions are equal.

How can you manipulate the equation so that the equation results in the Pythagorean theorem?


Converse of the Pythagorean Theorem
Lesson 4

**DIRECTIONS:** Complete the activities and examples that follow.

1. A student created these four triangles. However, two of these are incorrectly labeled as right triangles. Use the Pythagorean theorem to find which two of these four triangles are right triangles. Draw an X through the triangles that are not right triangles.

   **Hint:** The Pythagorean theorem only works for right triangles. In other triangles, $a^2 + b^2$ is not equal to $c^2$.

   ![Diagram of four triangles with measurements](image)

2. The Pythagorean theorem can also be used to find the value of a missing leg, given the value of one other leg and a hypotenuse.

   ![Diagram of a right triangle with measurements](image)

   **Start by writing the equation for the Pythagorean theorem (see p. 4.12.1).**

   **Substitute for the unknown amount.**

   $$a^2 = c^2 - b^2$$

   **Substitute and evaluate using numbers provided.**

   **State your answer both in exact form and using a calculator to approximate the answer.**
3. For the triangles below, use the information about the lengths of the hypotenuse and leg to find the other leg. Write your answers in the blocks provided above the triangle illustrations.

Answers:

<table>
<thead>
<tr>
<th>6.830</th>
<th>9.670</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.746</td>
<td>5.589</td>
</tr>
<tr>
<td></td>
<td>14.95</td>
</tr>
<tr>
<td></td>
<td>14.13</td>
</tr>
<tr>
<td></td>
<td>6.424</td>
</tr>
</tbody>
</table>

4. Many bridges employ a truss system that is made up of triangles.

Look at the illustration below. The truss system is 100 feet tall, the horizontal beam in the upper part of the bridge is 300 feet across, and the bottom beam is 500 feet long.

How long is each diagonal beam in the illustration?

Answer:
Similar Right Triangle Activity 1
Lesson 6

DIRECTIONS: Students should study this activity and inform the teacher of the angles they notice.
Eyes and Ears Protocol
Lesson 6

**DIRECTIONS:** Sit with a partner. While watching a video, one of you will pay attention to what you see, and the other will pay attention to what you hear. Take notes on your assigned role.

When you are done watching the video, the person who paid attention to what was seen should take two minutes to share what he or she wrote. The partner should write down what the “seer” shares. Then, the person who paid attention to what was heard should take two minutes to share what he or she wrote. The partner should write down what the “hearer” shares.

<table>
<thead>
<tr>
<th>What I Saw</th>
<th>What I Heard</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Similar Right Triangles Activity 2
Lesson 6

DIRECTIONS: Use your knowledge of Similar Right Triangle facts and ratios and solve the following problems. Show your work or explain how you arrive at your answer.

1. What is the value of $x$ in the triangle below?

![Diagram of a right triangle with sides 4 and 11 and unknown side $x$.]

2. What is the value of $x$ in the triangle below?

![Diagram of a right triangle with sides 4.6, 5, and 11 and unknown side $x$.]

3. The altitude to the hypotenuse of a right triangle divides the hypotenuse into segments whose lengths are 12 and 48. What is the length of the altitude to the nearest tenth?

4. In the diagram below, $\triangle RST$ is a 3—4—5 right triangle. The altitude $h$, to the hypotenuse has been drawn. Determine the length of $h$.

![Diagram of a 3—4—5 right triangle with altitude $h$.]
**Special Right Triangle Investigation 1**

**Lesson 7**

**DIRECTIONS:** In this investigation you will discover the relationship between the length of the leg and the length of the hypotenuse in a special right triangle.

1. Find the length of the hypotenuse in each of the triangles below. Make sure you simplify any radicals.

```
1
1

2
2

3
3
```

2. Complete the table below. Draw any additional triangles as needed.

<table>
<thead>
<tr>
<th>Length of each leg</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>...</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length of hypotenuse</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. What do you notice about the relationship between the length of the legs and the length of the hypotenuse? Is there a pattern? What is it?

4. Can you write a rule for this pattern? Example: In isosceles right triangles, if the legs have length 1, then the hypotenuse has length ______________.
Special Right Triangle Investigation 2
Lesson 7

OVERVIEW: In this investigation you will discover the relationship between the length of the leg and the length of the hypotenuse in a special right triangle.

1. Complete these tasks, using the illustration at right:
   a. Place point C somewhere along the horizontal line segment and label it C.
   b. Use your protractor to draw angle C with a measure of 30 degrees.
   c. Label the point where the angle crosses the vertical line as point A.
   d. Label the degree measures of each angle.
   e. Use the cm side to measure the hypotenuse and the short leg of your triangle, label these measures on your picture.
   f. How does the length of the short leg compare to the length of the hypotenuse?
   g. Now use the Pythagorean theorem to find the length of the long leg of the triangle.

2. Think about any pattern that you observed in Activity 1 or any rule that’s beginning to emerge. Apply them to the triangles below. Use your ruler, protractor, and the Pythagorean theorem to discover the lengths of the legs and the hypotenuse of each.

3. Write a rule or some rules regarding the relationships between the legs and the legs and the hypotenuse in this special (30-60-90) triangle.
OVERVIEW: In this activity you will discover the special relationships and ratios that exist between angles and the sides of right triangles.

Step 1: Draw 3 Triangles

a. **Triangle 1**—On graph paper, draw a right triangle with legs of 3 inches, 4 inches, and a hypotenuse of 5 inches.

b. **Triangle 2**—On the same graph paper, draw a second triangle, similar to the first but increased by a scale of 2 (Example: short side is 6 inches…).

c. **Triangle 3**—Draw a third triangle on the graph paper, similar to the first one, but increased by a scale of 3 (Example: short side is 9 inches…).

d. The three triangles you’ve just drawn are similar triangles because:

---

Step 2: Measure Angles and Sides of Triangles

For each of the 3 triangles:

- Label the short leg “a”, the longer leg “b”, and the hypotenuse “c”.
- Measure and record all the sides and angles.

**a. Triangle 1**—Complete this table with measurements for Triangle 1:

<table>
<thead>
<tr>
<th>Measure of side a</th>
<th>Measure of Side b</th>
<th>Measure of Side c</th>
<th>Side a/Side c</th>
<th>Side b/Side c</th>
<th>Side a/Side b</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**b. Triangle 2**—Complete this table with measurements for Triangle 2:

<table>
<thead>
<tr>
<th>Measure of side a</th>
<th>Measure of Side b</th>
<th>Measure of Side c</th>
<th>Side a/Side c</th>
<th>Side b/Side c</th>
<th>Side a/Side b</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Continued Next Side
Step 2, Continued:

**c. Triangle 3**—Complete this table with measurements for Triangle 3:

<table>
<thead>
<tr>
<th>Measure of side a</th>
<th>Measure of Side b</th>
<th>Measure of Side c</th>
<th>Side a / Side c</th>
<th>Side b / Side c</th>
<th>Side a / Side b</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Step 3: Answer These Questions

1. For all three triangles, compare this ratio:
   
<table>
<thead>
<tr>
<th>Ratio</th>
<th>Triangle 1</th>
<th>Triangle 2</th>
<th>Triangle 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Side a / Side c</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. For all three triangles, compare this ratio:
   
<table>
<thead>
<tr>
<th>Ratio</th>
<th>Triangle 1</th>
<th>Triangle 2</th>
<th>Triangle 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Side b / Side c</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. For all three triangles, compare this ratio:
   
<table>
<thead>
<tr>
<th>Ratio</th>
<th>Triangle 1</th>
<th>Triangle 2</th>
<th>Triangle 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Side a / Side b</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. If you created a fourth triangle, what could you predict about the ratios? Would they be the same, similar, or different?

5. Can we use the information about the ratios of leg lengths and relationships with angles to create rules about right triangles?
Right Triangle Ratios Lab
Lesson 9

OVERVIEW: Triangles are everywhere, and they are useful! Triangles are pivotal in the world of construction and design. Scientists and engineers compute and refer to trigonometric ratios of right triangles in their everyday calculations. In this lab we'll practice measuring the legs of right triangles, examining their relationships, and then make predictions and draw conclusions about their ratios.

1. Draw a 30° angle, and mark a point every 5 cm on a side as shown below. Draw perpendicular segments at each of the points.

![Diagram of a 30° angle with points marked every 5 cm on each side]

2. Measure the legs of each right triangle, and complete the table below.

<table>
<thead>
<tr>
<th>Triangle</th>
<th>Adjacent Leg</th>
<th>Opposite Leg</th>
<th>( \frac{\text{Opposite Leg}}{\text{Adjacent Leg}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \triangle ABC )</td>
<td>5 cm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \triangle ADE )</td>
<td>10 cm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \triangle AFG )</td>
<td>15 cm</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3. What is the relationship between $\triangle ABC$ and $\triangle ADE$?

4. Explain why the proportion $\frac{BC}{DE} = \frac{AC}{AE}$ and the proportion $\frac{BC}{AC} = \frac{DE}{AE}$ are true.

5. Predict the ratio of $\frac{FG}{AG}$. Why did you predict this value?

6. What can you conclude about the ratios of the legs of any right triangle with an angle measure of 30°?
**Ceiling Height Estimation Lab**

**Lesson 10**

**OVERVIEW:** Certain heights are difficult to calculate directly from ground level, such as the heights of tall buildings or the heights of tall ceilings. In this lab, we’ll estimate the height of a ceiling using trigonometric ratios and clinometers.

**Step 1: Clinometer Construction**

1. A clinometer is an instrument for measuring the angle of an incline (slope).
   To make your clinometer you will need:
   - a drinking straw
   - a protractor
   - tape
   - a coin
   - a piece of thread

   Use tape to attach the straw to the protractor, as shown in the diagram below. The thread with a coin at its end will help you measure the angle of inclination of an object as you look through the straw.

![Diagram of clinometer construction](image)

**Step 2: Partner Activities**

2. Find a partner, and have one person stand at a distance from the corner where the two walls meet the ceiling. Using a tape measure, have the other person measure the distance from the corner of the floor to the other person.

   Distance: \( d = \) ________________

   Continued →
3. Using the clinometer, one partner will view the corner of the ceiling through the drinking straw. Have the other partner record the angle of inclination when the corner of the ceiling is in sight through the straw.

   Angle: \( x = \) _______________

4. Label the right triangle with the value for \( d \) and \( x \).

   Which trigonometric ratio (sine, cosine, tangent) can be used to solve the height of the ceiling?

5. Write an equation that can be used to solve for the ceiling height, \( h \), and solve your equation to find an estimate for the ceiling height.

   Estimated height of the ceiling: \( h = \) _______________

6. How does your estimate compare to those of your fellow classmates? Why do you think they might differ?
The Extraordinary Race
Lesson 12

Project Packet

Contents:
Part A
  Race Course Map
  Overview
  Stages 1-5 Activities

Part B
  Culminating Activity

The unit of measurement used in this Performance Task is the metric system.

The Extraordinary Race was inspired by the “Sapphire Heist” by Diane Stobbe and Renee Handfield. It was revised by Martin Fechner, Don Symes, Tim Gartke and Jeremy Klassen, then adapted for this Guide. Permission for use of this Performance Task was granted by the originator, the Edmonton Public Schools, District 7, of Alberta, Canada.
The Extraordinary Race
Lesson 12

Race Course Map
The Extraordinary Race
Lesson 12

Part A

OVERVIEW: You are the leader of a team participating in a reality TV show. On this leg of the extraordinary race you need to journey from the starting point to the next checkpoint. Your job is to choose a path for your team to follow. Choose a path, complete the appropriate calculations, and put the path in a convincing presentation.

Each stage of this race to the ultimate prize requires you to determine an angle or a distance before continuing on.

Information about the race and its stages:

Stage 1. You are dropped onto a plateau by a helicopter.
Stage 2. You must race to the edge of the cliff using the path provided and use a zip line to cross the river to a point below the top of a cliff.
Stage 3. From here, you race to the next river crossing.
Stage 4. At this point you need to use the limited resources that you have been provided to get across the river AND up the cliff on the other side.
Stage 5. Once you are at the top of the cliff, it is a straight run to the checkpoint. Pace yourself, but try to get there first.

For each stage, you will need to include an explanation of your strategy for solving, given the problems.

The unit of measurement used in this Performance Task is the metric system.

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STAGE 1:

Choose a path (AB or AC) from the Race Course Map and determine the length of your chosen path.
The Extraordinary Race
Lesson 12

STAGE 2:

Determine the angle of depression of the zip line from different points (sites). At site B, the cliff height is 10m and the width of the river is 30m. At site C, the height of the cliff is 8m and the width is 23m.

Site B

Site C
STAGE 3:
For this leg of a journey, you need to determine the distance to the river. Use the information provided on the Race Course Map to determine the distance of DF or EG.
STAGE 4—FH Route:

At this point with your limited resources you need to be careful. You need to determine how much rope you need to connect to the top of the cliff on the far side.

For this stage two pictures of the river have been provided. One is from above and the other is from the shore. Use the information provided about the specific distances used and a protractor to determine the length straight from the shore to the top of the cliff.

Site F

From the river

From satellite the line segment drawn is 15.0m
STAGE 4—GI Route:

At this point with your limited resources you need to be careful. You need to determine how much rope you need to connect to the top of the cliff on the far side.

For this stage two pictures of the river have been provided. One is from above and the other is from the shore. Use the information provided about the specific distances used and a protractor to determine the length straight from the shore to the top of the cliff.

Site G

From the river

From satellite the line segment drawn is 39.0 m
STAGE 5:

Here you will need to run as fast as possible to get to the checkpoint. You need to determine the distance you will run so that you can pace yourself. Use the information provided in the Race Course Map on p. 2 of the Project Packet to determine the distance to the checkpoint (HJ or IJ).
Part B
OVERVIEW: From point J on the Race Course Map, you will need to get to point Z. Sources have told us that the trail is 3.16km, but you know that this does not get you to the end point. You will need to forge your way across the desert for an undisclosed distance. To determine the supplies you require, you must first figure out how far this is.

Distance from point J to point z ____________________________
Part A
Stage 1:

Go from A to C
\[ \cos 57^\circ = \frac{b}{2.1} \]
\[ 2.1 \cos 57^\circ = b \]
\[ b = 1.1 \text{ km} \]

Go from A to B
\[ \sin 57^\circ = \frac{c}{2.1} \]
\[ 2.1 \sin 57^\circ = c \]
\[ c = 1.8 \text{ km} \]

Stage 2:

Site B
\[ \tan \theta = \frac{10}{30} \]
\[ \theta = \tan^{-1} \left( \frac{10}{30} \right) \]
\[ \theta = 18^\circ \]
Stage 2, Continued:

Site C

\[ \tan \theta = \frac{8}{23} \]
\[ \theta = \tan^{-1} \left( \frac{8}{23} \right) \]
\[ \theta = 19^\circ \]

Stage 3:

Go from E to G
\[ \sin 52^\circ = \frac{x}{1.9} \]
\[ 1.9 \sin 52^\circ = x \]
\[ x = 1.5 km \]

Go from F to D
\[ \sin 46^\circ = \frac{x}{1.6} \]
\[ 1.6 \sin 46^\circ = x \]
\[ x = 1.2 km \]
Stage 4—FH Route:

Site F

From the river:

\[ \cos 20^\circ = \frac{x}{r} \]

\[ r = \frac{x}{\cos 20^\circ} \]

\[ r = \frac{32.2}{\cos 20^\circ} \]

\[ r = 34.2 \text{ m} \]

From satellite:

The line segment drawn is 15.0 m

\[ \tan 25^\circ = \frac{15}{x} \]

\[ x = \frac{15}{\tan 25^\circ} \]

\[ x = 32.1676 \]

\[ \therefore x \approx 32.2 \text{ m} \]
Stage 4—GI Route:

From the river:

\[
\cos 14^\circ = \frac{x}{r} \\
r = \frac{x}{\cos 14^\circ} \\
r = \frac{53.7}{\cos 14^\circ} \\
r = 55.3 m
\]

From satellite:

the line segment drawn is 39.0 m

\[
\tan 54^\circ = \frac{x}{39.0} \\
39.0 \tan 54^\circ = x \\
x = 53.7 m
\]
Stage 5:

From J TO I

\[ 1.0^2 + 1.8^2 = x^2 \]

\[ x = \sqrt{1.0^2 + 1.8^2} \]

\[ x = 2.1 \text{ km} \]

From H to J

\[ x^2 = 1.7^2 + 1.0^2 \]

\[ x = \sqrt{1.7^2 + 1.0^2} \]

\[ x = 2.0 \text{ km} \]

Part B:

\[ \tan \theta = \frac{1.8}{1.0} \]

\[ q = \tan^{-1}\left(\frac{1.8}{1.0}\right) \]

\[ q = 61^\circ \]

\[ \tan \theta = \frac{x}{3.16} \]

\[ 3.16 \tan \theta = x \]

\[ 5.7 = x \]

\[ \therefore x \approx 5.7 \text{ km} \]
Surface Area and Volume

TOPIC SEASON: Geometry—Measurement

This unit is designed for use in short-term programs.
Sections maybe adapted for long-term settings

Unit Designers: M. Bussiere, K. Chace, R. Dubuisson, and K. Miele

Introduction

Our students will often encounter real-world problems dealing with surface area and volume as they become young adults.

- How much paint will they need to purchase to paint the rooms in their homes?
- How much food is actually held inside the containers of food or liquids that they will purchase at the grocery store?
- What shape container will best hold and ship products?

The Spring Season of geometry asks students to use their mathematical reasoning to explore these questions in order to ensure that they can answer questions such as these.

The “Surface Area and Volume” unit is designed to come second in the Spring Season of Measurement and will take about seven days to complete, with the option of making the unit a full two weeks by expanding the Performance Task. The unit asks students to think about the questions:

- How can we apply volume formulas to real-life situations?
- What are the various ways to maximize the volume of a box?

“Students will use many discovery methods to learn the skills and knowledge set forth in this unit instead of relying on direct instruction from their teachers.”

Why would a business use one shape instead of another for packaging their products?

In order to explore these questions, students will learn how to maximize the volume of a box while using the smallest surface area possible. They will reimagine the packaging for a product that they could purchase and pitch a new packaging idea to the company that sells the product.

When students finish this unit of study, teachers will move on to a unit on circles and another on probability and measurement. In these units, students will continue to study the application of geometric concepts to real-world situations and will apply the knowledge and skills that they learned in the “Surface Area and Volume” unit to the units that follow.

The “Surface Area and Volume” unit primarily focuses on the standards:

G-GMD.3: Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.
G-MG.1: Use geometric shapes, their measures, and their properties to describe objects.
G-MG.3: Apply geometric methods to solve design problems.
To meet these standards throughout the unit, students will explore volume formulas and will discover connections among the various formulas.

The Performance Task requires students to apply their knowledge and skills to a real-world problem and to mathematically prove that their new packaging is superior to what is currently used.

To engage students with the standards and Essential Questions of the unit, this guide gives teachers many hands-on ideas, such as building and measuring three-dimensional objects, to show students how to measure for volume and how to maximize the volume of a container. Students will use many discovery methods to learn the skills and knowledge set forth in this unit instead of relying on direct instruction from their teachers. Students will also be engaged in the Performance Task and reading about a Consumer Reports article that asks them to think about why manufacturers package and ship their goods the way that they do.

In order to be successful in this unit of study, students must have an understanding of how to find the area of two-dimensional shapes. They should also understand terminology such as radius, height, length, and width so that teachers and students are using a common language when discussing how to find surface area and volume.

One challenge that this unit could present is using graphing calculators or online graphing calculator websites effectively. Teachers should be sure to give students enough time to learn through trial and error to use the calculator and to set appropriate windows to see their graphs. The teacher should allow students time to discover this important skill instead of telling students the appropriate windows to set for their graphs.

For adaptation ideas for this unit, see p. 5.11.3 on the right.
### Measurement: Surface Area and Volume
Adapting This Short-Term Unit for Long-Term Programs

<table>
<thead>
<tr>
<th>Plan 1 (Short)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPRING SEASON—<em>Surface Area and Volume</em>: Short-Term Programs</td>
</tr>
<tr>
<td>MONDAY</td>
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<tr>
<td>Week 1</td>
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<tr>
<td>Week 2</td>
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<table>
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<tr>
<th>Plan 2 (Long)</th>
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</thead>
<tbody>
<tr>
<td>SPRING SEASON—<em>Surface Area and Volume</em>: Long-Term Programs</td>
</tr>
<tr>
<td>MONDAY</td>
</tr>
<tr>
<td>Week 1</td>
</tr>
</tbody>
</table>

For the short-term program plan, the unit will stay as written (Plan 1).

For long-term programs (Plan 2), the teacher will have two extra days to work with students on solving equations for volume, surface area, and lateral surface area. The **Extension** that follows Lesson 3 provides time for students to practice the equations that they discover in Lesson 3. The **Review** day that follows allows teachers to review with students how to find area, surface area, and lateral surface area.

The long-term program plan allows for an extra day for the Lesson 7 Performance Task. Teachers can do the extension activity for the Performance Task by having students calculate the cost that the new packaging could save the company. The teacher will give students a basic cost for each foot of material used in the original packaging of their product. Students will calculate how much material was used in the original packaging. They will then calculate how much their new packaging requires. They can calculate how much the company would save on each sale, then research how much the company could save over the course of a year based on sales. This cost analysis would be included in the final presentation.

Lesson 1, “Finding Surface Area” is a review lesson that can be added by the teacher to the beginning of either the short- or long-term plan if it is determined that it would be needed by the student(s).
### Emphasized Standards *(High School Level)*

#### GEOMETRY

- **G-GMD.3:** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.
- **G-MG.1:** Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).
- **G-MG.3:** Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

#### Essential Questions *(Open-ended questions that lead to deeper thinking and understanding)*

- What are various ways to maximize the volume of a box?
- Why would a business use one shape instead of another for packaging their products?
- How can we apply volume formulas to real-life situations?

#### Transfer Goals *(How will students apply their learning to other content and contexts?)*

Students will apply their understanding of mathematical formulas to calculate the volume of real-world objects. Students will prove that certain shapes work better for different tasks.

Students will apply their understanding of volume and surface area to illustrate various ways to package products and learn why some are more appropriate for use in certain applications than others.

Students will apply their understanding of volume formulas to find the volume of real-world objects.
### Vocabulary:

- Volume formulas
- Right prism
- Rectangular prism
- Cylinder
- Radius
- Area
- Triangular prism
- Pentagonal prism

- Volume formulas for:
  - Cubes
  - Right rectangular prisms
  - Pyramids
  - Spheres

### Area

- Surface area
- Lateral surface area

### Volume

- Graph
- Domain
- Maximize

### Desired Results

<table>
<thead>
<tr>
<th>Students should know...</th>
<th>understand...</th>
<th>and be able to...</th>
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</thead>
<tbody>
<tr>
<td>Vocabulary:</td>
<td></td>
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</tr>
<tr>
<td>Volume formulas</td>
<td>The volume of various three dimensional objects can be calculated using formulas.</td>
<td>Calculate the volume of various three dimensional objects.</td>
</tr>
<tr>
<td>Right prism</td>
<td></td>
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<tr>
<td>Rectangular prism</td>
<td></td>
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<tr>
<td>Cylinder</td>
<td></td>
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<tr>
<td>Radius</td>
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<tr>
<td>Area</td>
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<tr>
<td>Triangular prism</td>
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<tr>
<td>Pentagonal prism</td>
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<tr>
<td>Volume formulas for:</td>
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<tr>
<td>Cubes</td>
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<tr>
<td>Right rectangular prisms</td>
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<tr>
<td>Pyramids</td>
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<td></td>
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<tr>
<td>Spheres</td>
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</tbody>
</table>

- Creating a chart showing data collection from various prism measurements, surface area, and volume.

- Objects with the same surface area can have different volumes.
  - Graphing equations can help us determine the maximum volume of a prism.
  - Prepare a presentation that explains to a business the best packaging for them.
  - Maximize the volume of a box made from a single piece of material.
Assessment Evidence

Quality questions raised and tasks designed to meet the needs of all learners

Performance Task and Summative Assessment (see pp. 5.12.16-5.12.17)

Aligning with Massachusetts standards

Students will imagine that they are recreating the packaging of a product, and they will pitch their idea in a presentation to executives of the product’s company. Students will think about the best packaging that can be created to hold the most product with the smallest surface area.

The students will prove to the executives that they will save money by changing the packaging because they will use less materials to package the product and will be able to ship it more efficiently. They will compare the new package to the previous package in order to show executives how much packaging material they will save. They will create graphs that show the maximum volume in order to prove that the new package is best.

Pre-Assessments (see pp. 5.12.7-5.12.10)

Discovering student prior knowledge and experience

Lesson 1: Review (if students need review) and Assessment
Lesson 2: Creating a Container Practice and Application Activity, Exit Ticket

Formative Assessments (see pp. 5.12.10-5.12.15)

Monitoring student progress through the unit

Lesson 3: Volume Formula Discovery and Explanation
Lesson 4: Chart and Pattern Explanation
Lesson 5: Shape and Volume Exit Ticket
Lesson 6: Maximizing Volume of a Box Activity Sheet and Exit Ticket

For Empower Your Future Connections, see p. 5.13.1
Access for All

Considering principles of Universal Design for Learning (UDL), Positive Youth Development/Culturally Responsive Practice (PYD/CRP), differentiation, technology integration, arts integration, and accommodations and modifications

Multiple Means of Engagement

This is the why of learning. It is what makes students engage or disengage. Throughout the unit plan, the student may be provided with as many choices in the level of challenge and complexity as possible in order to recruit and sustain engagement. For example, the teacher will encourage and support students in setting their own personal, academic, and behavioral goals. The teacher will use many strategies to guide students, including reminders, guides, rubrics, checklists, and prompts among other things that focus students on self-regulatory goals. Student tasks will be varied, allowing for active participation, exploration, and experimentation. The teacher will provide differentiated models, scaffolds, and feedback, as well as content material that is culturally relevant and responsive to students’ backgrounds. Most important is that teachers design assignments and tasks with authentic outcomes, and that they are purposeful and convey meaning to real audiences.

The lessons in this unit are designed to show students the real-world applications of the skills that they are taught, such as measuring, graphing, and solving equations. The Performance Task in this unit is designed to allow students to use their knowledge of volume and surface area to create a new packaging container for a product.

Students will be expected to “discover” mathematical concepts and equations rather than memorizing equations that the teacher dictates. Students will work independently and with peers in order to learn new concepts such as how we find the volume of prisms.

Students will be able to show their understanding verbally, through writing, and through drawings.

All students are encouraged to demonstrate their reasoning, regardless if they come to the “correct” answer. A problem does not have to be done correctly to be worthy of discussion; in fact, it can be helpful to the class to see a problem done incorrectly so that they understand misconceptions.

Accommodations intended to adjust this unit’s learning and language objectives, Transfer Goals, level of performance and/or content may be necessary for students with mandated specially designed instruction described in their Individualized Education Programs (IEPs). Modifications may be necessary for students with limited English proficiency.

Multiple Means of Representation

This is the what of learning. There are many pathways to conveying information to students. Throughout the unit, the teacher will provide information and materials in several modalities such as diagrams, vocabulary handouts, word walls, posters, and charts with formulas for calculations; and models, videos, and audio for text. The teacher will also demonstrate concepts through hands-on activities such as measuring the volume of various containers.
The way information is displayed should vary, including size of text, images, graphs, tables or other visual content. Where possible, written transcripts for videos and auditory content should be provided. Teachers will use videos to show students real-world connections to the mathematical skills they are studying. Information should be chunked into smaller elements, and complexity of questions can be adjusted based on prior knowledge competency. Reference sheets for examples, notes, vocabulary, and definitions can be differentiated for content.

**Multiple Means of Action and Expression**

This is the *how* of learning. In learning activities, students will be provided options for demonstrating what they know and can do. Students will have options of writing or verbalizing their learning. Students will have access to assistive technology. For example, students will have access to word processors with grammar checks, word prediction, and spell checkers. The teacher will also break down long-term goals into short-term reachable goals.

Performance Tasks can be differentiated by content, process, and/or product to address various learner profiles. Students will have a choice for the Performance Task as to how they want to show their mathematical reasoning to their classmates. They can create a poster or PowerPoint presentation when they present their new packaging to the class. Students should be given high- and low-tech options to compose in multiple media, such as text, speech, drawing, or illustration. Students will use websites to aid them in comprehending mathematical concepts and will use drawings and equations to solve problems. They will also be able to engage in hands-on activities by building and measuring prisms. Students can use graphic organizers, such as the T-charts in Lesson 3, Excel spreadsheets to organize data (Lesson 4), drawings by hand (Lessons 3, 4, 5, 6 and the Performance Task), checklists, index cards, or sticky notes for vocabulary, to better understand and demonstrate comprehension of the material. Opportunities for collaboration and whole-class discussion should be provided as needed.

**Literacy and Numeracy Across Content Areas**

**Reading**

Students will read and follow the instructions and information for the assignments. Students will also read about applications of volume in the real world through an ABC News report about an informational *Consumer Reports* article. They will analyze the information provided in the story and create a pro/con list based on the reading. They will read charts, graphs, and tables to solve problems or create workable solutions.

**Writing**

Students will engage in writing activities through “Do Now” activities or “Exit Tickets.” They will write to explain their reasoning and prove their answers through writing.

**Speaking and Listening**

Students will speak with their teacher and classmates in order to complete all of the assignments in this unit. Students will share their reasoning with their classmates and build on the ideas of their classmates to
clarify their own thinking. For the Performance Task, students will create a presentation that pitches a new package to a company. They will be persuasive and use mathematical reasoning to convince their audience that the packaging of a product should be changed.

**Language**

Students will use content-specific mathematical vocabulary to explain the concepts discussed in this unit.

**Numeracy**

Students will solve equations for surface area and volume and apply the equations to real-world situations. Students will create graphs and read them to maximize the volume of a prism.

**Resources** (in order of appearance by type)

**Print**


**Websites**

Lesson 1


Lesson 3


Lesson 4


Lesson 5


Lesson 6

Materials
Paper, tape, ruler, scissors

Lesson 4: Volume of a Cube, Rectangular Prism, and Cylinder Activity Sheet p. 5.14.1
Lesson 6: Maximizing the Volume of a Box Activity Sheet p. 5.14.2
Lesson 7: Sample Rubric for Performance Task Teacher Resource p. 5.14.3
Lesson 7: Performance Task Rubric (blank) Activity Sheet Google Drive (DYS/SEIS educators only)

PREREQUISITES: Math skills needed for this unit

Surface Area and Volume is the second unit in the Spring Season of geometry. Lesson 1 offers the teacher the ability to gauge student readiness for the subsequent lessons in the unit and may be deleted if the teacher decides it is not needed. The following skills will be needed for students to successfully complete this unit.

Students should know:

- How to find the surface area of various 2-dimensional shapes
- This terminology: base, radius, height, length, width
- How to use formulas to calculate the area of geometric shapes including: triangles, squares, and other parallelograms

Outline of Lessons
Introductory, Instructional, and Culminating tasks and activities to support achievement of learning objectives

INTRODUCTORY LESSONS

Stimulate interest, assess prior knowledge, connect to new information

Note: Teachers should vet all videos included in this unit according to program standards and create templates or graphic organizers for students to monitor their comprehension of the material.

Lesson 1
Finding Surface Area

Goal
Students will review how to find the surface area of an object and review unit vocabulary.
**Do Now** (time: 5 minutes)
The teacher will give students index cards as they enter the room. Each index card will have a different prism on it (cylinder, pyramid, cube, rectangular prism). The teacher will ask students to identify the type of prism and label the prism with the words width, length, height, radius, and base. The teacher will tell students that depending on the object they have, they might not use all of these words.

**Hook** (time: 10 minutes)
Students will come to the front of the room and share their index card with the class. The teacher will correct any misconceptions that the students have about vocabulary. Students will create larger representations of their prisms, label them correctly, and post them around the room.

**Presentation** (time: 15 minutes)
The teacher will review how to find the surface area of two dimensional (2D) shapes by drawing a rectangle on the board and showing students that they can find the area by multiplying the length by the height.

The teacher will then use the *Braining Camp* website to show students how they can find the surface area of prisms. The website shows students the nets of various prisms and explains how to calculate the surface area.

**Practice and Application** (time: 10 minutes)
As a class, or individually, students will practice finding surface area using the “Surface Area: Practice with Math Games” activity.

The teacher will help students who are struggling with the problems. Students should use the folded paper they created in the presentation to remind themselves of the equations they need to use to find the surface area of the prisms.

**Review and Assessment** (time: 15 minutes)
Students will answer the questions on *Braining Camp*.

When they finish answering the ten questions, they will submit their answers. The website will tell students which questions they answered incorrectly. Students will make note of the questions they answered incorrectly and answer the questions again, figuring out why they answered the questions incorrectly. Students will explain, either verbally or in writing, one mistake that they made, why they think they made the mistake, and how they came to find the correct answer.

---

**SEE:**
- Braining Camp Lesson: Surface Area
  - www.brainingcamp.com/legacy/content/concepts/surface-area/lesson.php
- What is the Surface Area?
  - www.mathgames.com/skill/7.145-surface-area
- Surface Area Questions
  - www.brainingcamp.com/legacy/content/concepts/surface-area/questions.php
Lesson 2

Experimenting with Volume

Goal
Students will create a container that holds the maximum volume that it can, using a given-sized piece of paper.

Do Now (time: 5 minutes)
The teacher will ask students to think about three-dimensional objects that they see daily:

What three-dimensional shapes do you see in the world around you that are used to store things?
What types of three-dimensional shapes do you see in nature?

Students might mention that tree trunks look like cylinders, the earth is a sphere, and so on. Students will brainstorm a list individually and then share their responses with the class.

Hook (time: 5 minutes)
The teacher will show students a few different shaped containers that are used to store and ship food. For example, the teacher might have a box of microwavable popcorn and a Pringles chips container.
The teacher will ask students:

Why do companies use different shaped boxes to store and ship their goods?
What are the benefits of using different shapes and sizes?

Students will discuss their answers with the class. They might mention that the containers are created to fit the shape of the object that they are selling. They might mention that some containers stack easier in boxes to ship. The teacher should make a list of these reasons and keep them posted so that students can refer back to them when they work on the Performance Task.

Presentation (time: 10 minutes)
The teacher will explain to students that at the end of the unit, they will be creating a presentation for a company that will inform them about what shape and size container they should use to ship and store their product. The teacher will explain that before students begin to work on their final projects, they will create a rubric together that will be used to assess the presentation and project. This way, all students will understand the expectations of the assignment. The teacher will reiterate that students will need to support their findings using mathematical reasoning.

Students will explore how many different shaped containers and different sized containers they can create with the same size paper. The goal is for students to maximize the volume of their container. Students will not understand how to do this mathematically; they are using trial and error here. The teacher will tell students that they all have the same surface area to work with to create their container and will remind students what surface area is and how to calculate it. The teacher will explain that their goal is to create a container that holds more than any other container could hold that was made with the same sized paper and, therefore, the same surface area. The teacher will hand out paper, tape, a ruler, and scissors, if possible.

Practice and Application (time: 25 minutes)
The teacher will demonstrate how a container’s volume can be measured using cereal and a measuring cup.
(Puffed Rice works well for this purpose). Students will work to create their containers. The teacher will circulate around the room to help students think about what they are creating. The teacher will remind students that there are many shapes that they can make with their pieces of paper. As students create their containers, the teacher will fill them with the cereal and have the students use a measuring cup to measure how much cereal fits in the container. Students will record the type of container they created, the height and width of the container, and the shape of the container so that they can analyze trends. This information should be posted somewhere for all students to see.

**Review and Assessment** (time: 10 minutes)

Students will review the different types of containers created by their classmates and how much they held. As an Exit Ticket, students will write down any trends that they see in the types of containers that hold the most volume. They will explain, in writing, what shape is best for shipping the cereal that was poured into their containers. The teacher will collect these Exit Tickets to review and assess what students have learned and what trends they noticed in the creation of these containers. The Exit Tickets will give the teacher an idea of what students already know about surface area and volume.

**INSTRUCTIONAL LESSONS**

*Build upon background knowledge, make meaning of content, incorporate ongoing Formative Assessments*

**Lesson 3**

Discovering Volume Equations

**Goal**

Students will discover various equations to solve for the volume of three-dimensional shapes.

**Do Now** (time: 5 minutes)

Students will write down one thing that they remember from yesterday’s class about the relationship between volume and shape. They will share their responses with the class to make a master list of things that they remember. The teacher will use this time to correct any misconceptions that students have.

**Hook** (time: 5 minutes)

The teacher will have three different shaped containers in the front of the room. The teacher will ask students:

Which container will hold more water? Why do you think that is true?

The teacher might want to purposely find containers that are deceptive as to how much they will hold. This may include containers with thick walls or bottoms. Students will discuss why they think a certain container will hold more water and will try to justify their answers with observational evidence and information that they remember from the previous class.
**Presentation** (time: 5 minutes)
The teacher will use a measuring cup to fill the containers with water to show the students how much each container holds. The teacher will ask students if they are surprised by the results. Ask students:

How could we find the volume of an object and compare the volume of these containers if we couldn’t fill them with water?

Allow students to discuss some possible solutions to the question. The teacher will tell the students that their goal today is to discover formulas that will allow them to mathematically find the volume of different shaped objects.

**Practice and Application** (time: 25 minutes)
The teacher will draw three shapes on the board, label the width, height, length, and/or radius and tell students the volume of the object. The students will work with partners to discover the mathematical formula that will always find the volume of the shape.

*One object will be a cube.*
The teacher will label the sides 3, 3, and 3, and tell students that the volume is 27.

*Another object will be a cylinder.*
The teacher will tell students that the radius is 3 and the height is 5. The volume is 45.

*The last object is a pyramid.*
The teacher will label the width 3, the length 3, and the height 5, and tell students that the area is 15.

Students might find it easier to use manipulatives to explore these formulas.

The teacher can give students small cube blocks or create paper cubes to allow students to build the 3x3x3 cube. This will help them find an equation for volume.

**Review and Assessment** (time: 15 minutes)
Students will share their answers with the class and explain how they came up with the equations that they did. Even if students did not discover the correct equation, they should explain their thinking and how they tried to find the answer.

The teacher will then share with students the equations for volume. Students should create a T-charts with the equations on one side and a drawing of the shape on the other. The teacher will post these formulas around the classroom so that students are constantly seeing them.

The equations that students should know are:

<table>
<thead>
<tr>
<th>Shape</th>
<th>Formula</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube</td>
<td>( V = s^3 )</td>
<td>( s ) is the length of the side</td>
</tr>
<tr>
<td>Right Rectangular Prism</td>
<td>( V = lwh )</td>
<td>( l ) is the length, ( w ) is the width, and ( h ) is the height</td>
</tr>
<tr>
<td>Prism or Cylinder</td>
<td>( V = ah )</td>
<td>( a ) is the area of the base, ( h ) is the height</td>
</tr>
<tr>
<td>Pyramid or Cone</td>
<td>( V = \frac{1}{3} ah )</td>
<td>( a ) is the area of the base, ( h ) is the height</td>
</tr>
<tr>
<td>Sphere</td>
<td>( V = \frac{4}{3} \pi r^2 )</td>
<td>( r ) is the radius</td>
</tr>
<tr>
<td>Pentagonal Prism</td>
<td>( V = \frac{1}{4} \sqrt{5(5 + 2\sqrt{5})} a^2 h )</td>
<td>( a ) is the base edge, ( h ) is the height</td>
</tr>
</tbody>
</table>
Extension
Students can practice using volume formulas by using this resource on the Annenberg Learner website.

SEE: Interactives: Geometry 3D Shapes
www.learner.org/interactives/geometry/area_volume.html

The teacher and students can also visit the Massachusetts Comprehensive Assessment System (MCAS) webpage to view or download the Grade 10 Mathematics Reference Sheet.

SEE: Grade 10 Mathematics Reference Sheet
www.doe.mass.edu/mcas/tdd/math.html?section=resources

Lesson 4
Discovering Patterns for Volume Formulas

Goal
Students will discover patterns that exist in the volume formulas of various prisms.

Do Now (time: 5 minutes)
The teacher will project a cube, a rectangular prism, and a cylinder on the board or the teacher can use the “Volume of a Cube, Rectangular Prism, and Cylinder” Activity Sheet located on p. 5.14.1 in the Supplement. Students will use the formulas that they learned yesterday to find the volume of each.

Hook (time: 5 minutes)
The teacher will ask students:

Do you think that patterns exist in volume formulas? Why or why not?

Encourage students to discuss in pairs and provide scaffolded support as necessary.

Presentation (time: 10 minutes)
The teacher will show students the video “Volume and Surface Area” to remind students how to find surface area of prisms and to show students a visual of how three-dimensional objects can be “unfolded” to find surface area. Options for a graphic organizer may be provided so students can write or draw to reinforce comprehension.

SEE: Volume and Surface Area
www.youtube.com/watch?v=vZPeEBigUpk

The teacher will give the students rulers and will provide students with numerous prisms that have different bases. The teacher will tell students that they will collect various pieces of information from each prism. Students can use Excel, or the teacher can provide graph paper, to record the height of each prism, the area of each base, the circumference of each base, the lateral surface area of each prism, the total surface area of each prism, and the volume of each prism. The teacher will tell students to keep in mind the goal is to find patterns in the formulas.

The teacher will review how to calculate the lateral surface area and the total surface area of prisms. An
MCAS review sheet can be given to students so that they have a handout to look at to remind them of the formulas.

SEE: MCAS Grade 10 Mathematics Reference Sheet (PDF)

Practice and Application (time: 25 minutes)
Students will measure the objects and record their data. To organize and record their data, students will create an Excel spreadsheet or other organizer the teacher may choose. If using Excel, they should list the types of prisms they are studying in a list in column A (starting in Row 2), then list the types of data that they are collecting for each prism in individual cells in Row 1 (Starting in column B). Once they record the data, they will talk with their partners and attempt to find patterns in the formulas. The teacher will help students who are struggling by asking them questions such as:

- What are you multiplying to find the surface area for each shape?
- How does that relate to volume?
- Why is finding the base measurement important?

Review and Assessment (time: 10 minutes)
Students will share their findings with the class. Students will explain the process that they went through to find the patterns that exist. They will start by sharing what they noticed about the volume formulas, then they will explain how they discovered the patterns. Students will write down one thing that they noticed about volume formulas and surface area of various prisms and turn it in as an Exit Ticket.

Lesson 5
Discovering How Shape Impacts Volume

Goal
Students will discover how the shape of a prism impacts its volume.

Do Now (time: 5 minutes)
The teacher will post the question on the board:

- How does the shape of an object impact its volume?

Students will do a think/pair/share by writing down their thoughts before sharing them with partners and then sharing them with the class.

Hook (time: 15 minutes)
The teacher will give all students an 8 ½ x 11-inch piece of paper and ask them to create a prism. Students will line up their prisms in the front of the room and discuss which has the biggest volume.

Following the discussion, the teacher should review the vocabulary cards or word walls with the students and update as necessary.
Presentation (time: 10 minutes)
Using one of the student's prisms, the teacher will review how to find the dimensions and compute the volume. The students should follow along and find the volume of their own prisms. Each student should show his/her prism and explain its volume. The teacher will explain to students that they will be making a minimum of one more prism to decide what shape has the largest volume. (This will allow students to begin to decide if a rectangular box will hold more volume than a square box or a cylinder. They might even try to create a triangular prism to measure. The more variety the better, as long as all the shapes are right prisms and students can measure the appropriate lengths and calculate their volume.)

Practice and Application (time: 15 minutes)
In pairs or small groups, students will create at least one more prism that has a larger (or at least different) volume than their original. Students will use the 3-D Rectangular Prism Maker to create new prisms. As they change the width, height, and length, they will see how the surface area and volume of the prism changes. If students want to create different shaped prisms, they can sketch the prism or create it with paper. They will record the dimensions and volume on a separate piece of paper.

Review and Assessment (time: 10 minutes)
The class will report on the shape their group found to have the largest volume. The class will discuss any patterns they see. As an Exit Ticket, students will answer the questions:

- What did you learn today about volume of different shapes?
- What shape do you think will give the greatest volume?

Lesson 6
Maximizing Volume Graphically

Goal
Students will graph equations to find the maximum volume of a box.

Do Now (time: 10 minutes)
The teacher will put the following equations on the board:

\[ y = 2x - 15 \quad y = -0.5x^2 + 2x + 3 \quad y = x^2 - 2x^2 + 5 \]

Students will use an online graphing calculator to graph these equations. They must make sure that the entire graph shows up in the window of their calculators.

SEE: Equation Explorer
https://kevinmehall.net/p/equationexplorer/

Note: The teacher might need to provide additional time for students to explore the Equation Explorer graphing calculator website. The website is fairly self-explanatory. Students simply click on the “add
equation” box and type in the equations that the teacher provided. Students need to know that they have to use the ^ symbol in order to put in exponents, but otherwise, they should be able to enter the equations without any problems. The website automatically graphs the equations for the students.

**Hook** (time: 5 minutes)
The teacher will ask students:

Until now, how have we been trying to figure out how to best maximize the volume of a prism?

How do we know if we have actually created a prism with the most volume possible?

Students will say that they have been measuring objects that they have created to find the volume of the objects and they have been using trial and error to make a prism with the highest volume.

**Presentation** (time: 15 minutes)
The teacher will review the Do Now, specifically looking for graphs with a good window. Students will need to understand how to set an appropriate window so that they can see the whole graph. The teacher should model how to set up a window. After the students have watched the teacher model, they should be given appropriate time to practice how one determines a good window for an equation. Do not give this window to the students; an important part of learning to using the graphing calculator is to learn to “set a window.” The teacher should monitor the students’ ability to “set a window” and provide scaffolded support to those who need additional guidance. It is important that they understand how to set the window now because future graphs that they create will have more difficult windows to set.

The teacher will hand out the “Maximizing the Volume of a Box” Activity Sheet to students (p. 5.14.2 of the Supplement) and they will read it together. The teacher will ask students to sketch out the problem as a first step. This is an important step in understanding the problem.

Then the teacher will ask students what the formula is for volume of a rectangular prism ($V = l \times w \times h$).

Using the Activity Sheet and their sketches, the teacher will ask students to:

Identify the length, width, and height of the box that is created when the corners are cut out.

Label the sides as follows:

one side is $20 - 2x$, another side is $16 - 2x$ and the third side is $x$.

This will give them a formula for Volume in terms of $x$: $V(x) = x(20 - 2x)(16 - 2x)$. A good window for this is $[-1, 15]$ for the x’s and $[-50, 500]$ for the y’s.

**Practice and Application** (time: 20 minutes)
Once the students all have a good graph on their calculators, their task is to find the point which gives them the maximum volume and finish question 1. They should then do the second problem, then write and hand in their reflections. (See the “Maximizing the Volume of a Box” Activity Sheet on p. 5.14.2 of the Supplement.)

**Review and Assessment** (time: 5 minutes)
As an Exit Ticket, students should explain what the best shape is to maximize volume. They should write their explanation on a piece of paper and hand it in for the teacher to review.
CULMINATING LESSON

Includes the Performance Task, i.e., Summative Assessment—measuring the achievement of learning objectives

Lesson 7

Creating Improved Product Packaging (2 days)

Goal
Students will create a new and improved container to store and ship a product that maximizes the volume of a given amount of packaging material.

Do Now (time: 5 minutes)
Students will create a list of products that they could purchase for which they think they could create a better shipping and storage container. Students will share their lists with the class in order to create a master list of ideas. This list should include awkwardly-shaped products.

Hook (time: 15 minutes)
The teacher will read students an ABC News report on a Consumer Reports article about product packages.

SEE: Air to Spare: Why Are Product Bags, Boxes Not Full?

After reading the report, students will create a pro/con list that lists reasons why companies might want to use extra packaging material or less packaging material for their products. Students will share their lists with the class.

Presentation (time: 15 minutes)
The teacher will explain to students that their goal is to create a presentation to executives of the company whose product they are recreating the packaging for. They want to think about the best packaging that can be created to hold the most product with the smallest surface area.

The teacher will create a rubric with the class that will assess the students’ presentations. The "Sample Rubric for Performance Task" on p. 5.14.3 of the Supplement provides a completed example for teachers. A blank Performance Task Rubric is available on Google Drive.

SEE: Performance Task Rubric (blank)

The teacher will lead students through the discussion of what they will need to include in this presentation to convince the executives to change their packaging. The students should say that they need to prove to the executives that they will save money by changing the packaging because they will use less materials to package the product and will be able to ship it more efficiently. Students will need to compare the new package to the previous package in order to show executives how much packaging material they will save.

They will want to create graphs that show the maximum volume in order to prove that the new package is best.

Note: Students likely can find the dimensions of their product’s box online. The teacher might need to help students find this information if students do not have access to searching the internet.
Practice and Application (time: 50 minutes | Day 1—20 minutes, Day 2—30 minutes)
The students will create the new and improved container to hold the product whose packaging they are reimagining. The teacher will provide students with paper, tape, and scissors if possible. Students will create the new package and prove mathematically that their new container is a more efficient and practical way to package and ship their product. They will need to create a presentation that will prove this to their audience—the executives of the company that owns the product.

In their presentations, students will need to show the executives how much packaging material they will save by using the new and improved design. They will also show how much easier it will be to pack and ship the product with the new design that they have created.

Review and Assessment (time: 25 minutes)
Students will practice their presentations with partners and will score themselves on the rubric that they helped to create. If students realize that they need to revise part of their presentation, the teacher will give the students time to do so.

Students will present their presentation to the class, who will pretend to be the executives of the company. Students will give the presenters feedback that tells them why they were or were not convinced to change the packaging of their product.

Extension
The teacher could extend this project by having students calculate the cost that the new packaging could save the company. The teacher will give students a basic cost for each foot of material used in the original packaging of their product. Students will calculate how much material was used in the original packaging. They will then calculate how much their new packaging requires. They can calculate how much the company would save on each sale, then research how much the company could save over the course of a year based on sales.
POST–UNIT REFLECTION
On meeting the Learning and Language objectives
Connections to Empower Your Future
UNIT: Surface Area and Volume

Future Ready Connections

This unit is interactive and depends on students actively engaging with the curriculum and talking and working collaboratively.

Teachers are encouraged to use the Future Ready Rubric to evaluate students’ growth in developing communication skills, demonstrating accountability, and having the ability to take initiative during these hands-on activities and projects. Youth have many opportunities to strengthen their communication and listening skills through group discussions and short presentations, and through the presentation of their Performance Task. Students will also communicate through writing in their Do Nows and Exit Tickets which can be evaluated for clarity, coherence, and critical thinking. Youth should also be evaluated for initiative and self-direction, especially when they must construct different objects to measure volume and when they create the Performance Task.

Teachers should reflect on whether or not youth stay on task without prompting and if they push themselves to thoroughly complete each activity, propose theories, answer their own questions, and create a detailed final product instead of only addressing the minimum required information. Teachers should encourage students to reflect on how they demonstrated growth and increased understanding throughout the unit, what they could do to further improve their skills and understanding, and how what they have learned is transferable to other situations and experiences.

Career Exploration Connections

Teachers can expand on the unit by making connections between the content and career exploration research. Students can research which career fields and jobs must understand, utilize, and plan for volume and surface area. Examples include painters, concrete and cement companies, real estate agents, storage companies, shipping companies, farmers, architects, and fashion designers. Students can use the MassCIS website to research jobs, career fields, industries, and programs of study associated with volume and surface area.

SEE: MassCIS
https://portal.masscis.intocareers.org

Teachers can also provide brief descriptions of jobs that use volume and surface area computations and ask students to brainstorm how these math concepts apply. For example, a teacher can explain to students that a fashion designer works with large pieces of fabric that will be cut down to make a garment and then ask students why fashion designers need to understand surface area. Fashion designers need to understand how many pieces of a garment they can cut from one piece of fabric. This is important because they don’t want to run out of fabric and they also don’t want to waste fabric by not laying out the pieces in a way to use up the least amount of surface area. For a hands on exercise, teachers can give students a sense of what it is like to be a fashion designer by giving students a piece of paper and several different shape pieces of colored paper and asking them to fit as many shapes onto the sheet of paper as they can.

“Students are active participants in both the activities and in the direction and pacing of their own learning.”
21st Century Workplace Skills

This unit provides many opportunities for teachers to name and emphasize the workplace skills that students are using and developing as they complete each task. Throughout the unit, students use different tools that may be required in a future workplace such as: Excel software, ruler, scissors, graphing calculator, and other online software (3D rectangular prism maker). Teachers can also point out the soft skills that students are practicing in the unit such as: communication, giving presentations, critical thinking, identifying pros and cons, and so on. By pointing out the skills required by the lesson, teachers can encourage youth to see their skill development and provide them examples to use when discussing their skills with future employers or interviewers.

PYD/CRP Connections

This unit embraces Positive Youth Development by encouraging students to be leaders in the classroom and to be active participants in their learning. Students are responsible for educating each other and working together to confirm conclusions. Youth have the opportunity to work with, support, and learn from each other’s discussions, presentations, and representations of data. They will both self-assess and provide constructive feedback. The responsibility is placed on each youth to demonstrate their own learning and support the learning of their classmates which allows for interpersonal skill development, leadership skill development, and positive relationships. The lesson also allows youth the chance to answer questions, get immediate feedback, learn from their mistakes, and then target areas for improvement. Students are active participants in both the activities and in the direction and pacing of their own learning. Teachers are encouraged to note these moments of leadership, interpersonal skill development, and positive relationships to their students so that students are consciously aware of their personal development.

This unit reflects Culturally Responsive Practice by using realistic situations such as the need to pick appropriate packaging for different products and the appropriate size containers for shipping an item. Teachers could expand on this concept by connecting the idea of container volume to familiar situations such as packing a suitcase, packing a backpack, or storing items in a closet or box.

Connecting math concepts to meaningful and purposeful contexts allows for authentic outcomes that youth can practice and apply independently in the future.

Another way that teachers can address Culturally Responsive Practice is to encourage students to reflect on their communication and mental processing methods and analyze the impact of those personal practices on how they designed their Performance Task. In the Performance Task, students will design a presentation to give to a company and it is important for them to reflect on how their own personal experiences, expectations for communication, and approach to teaching and learning influence the way they work with others. What would they do if the company didn’t understand or like the way they normally communicate or explain ideas? This reflection will allow youth to consider what they are bringing to the table and how to utilize their experiences and skills.
Volume of a Cube, Rectangular Prism, and Cylinder
Lesson 4

DIRECTIONS: Students should use formulas from the chart below to solve for volume.

<table>
<thead>
<tr>
<th>Volume Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V = s^3 )</td>
<td>( s ) is the length of the side.</td>
</tr>
<tr>
<td>( V = lwh )</td>
<td>( l ) is the length, ( w ) is the width, and ( h ) is the height.</td>
</tr>
<tr>
<td>( V = ah )</td>
<td>( a ) is the area of the base, ( h ) is the height.</td>
</tr>
</tbody>
</table>

Cube:
Side Length = 2.5 inches
Volume = ________________

Rectangular Prism:
Height = 4.2 inches
Length = 8.4 inches
Width = 3.1 inches
Volume = ________________

Cylinder:
Radius = 2.1 inches
Height = 5.7 inches
Volume = ________________
Maximizing the Volume of a Box
Lesson 6

DIRECTIONS:
Answer all questions below.
Show all your work and write answers in complete sentences. Reflect carefully on the last question, as this will be partially used in your final unit assessment.

1. You are making a rectangular box out of a 16-inch by 20-inch piece of cardboard. The box will be formed by making cuts of equal size in each corner and folding up the sides. You want the box to have the greatest volume possible.
   a. Write an equation for the volume of the box in terms of \( x \), the size of the equal cuts.
   To find the maximum volume, graph the volume function on a graphing calculator. What window did you use to “see” the whole graph including all maximum and minimum points, and \( x \)- and \( y \)-intercepts?
   b. How long should you make the cuts?
   c. What is the maximum volume?
   d. What will the dimensions of the finished box be?

2. Now change the size of the paper to the 8 x 11 paper you used earlier in the unit and redo each step.
   a. Write an equation, graph with a good window, and find the dimensions that will give you the maximum volume.

3. Write a reflection on how this assignment connects to the topics you have been working on in this unit.
Sample Rubric for Performance Task
Lesson 7

**Note:** This rubric is an example for teachers. Teachers will create a similar rubric with the students to ensure they understand what they are creating in the Performance Task. Familiar language should be used.


<table>
<thead>
<tr>
<th>Process of completing project</th>
<th>Advanced</th>
<th>Progressing</th>
<th>Beginning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student uses class time effectively, asks questions when necessary, and puts in the effort needed to complete task in the allotted amount of time.</td>
<td>Student mostly uses class time effectively and often asks questions when s/he is struggling.</td>
<td>Student does not use class time effectively and does not ask questions when s/he is unsure of how to complete the task.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mathematical explanation of why new packaging is better</th>
<th>Advanced</th>
<th>Progressing</th>
<th>Beginning</th>
</tr>
</thead>
<tbody>
<tr>
<td>The visual part of the presentation shows mathematical equations that clearly and accurately depict how much material could be saved if the company changes its packaging materials. All mathematical equations are solved accurately. Graphs are accurate and easy to read.</td>
<td>Student shows mathematical equations on his/her visual that explain how much packaging material can be saved with the new packaging. The equations are mostly accurate, but may contain minor errors. Visual is not as clear to the audience as those in the advanced category. Graphs are accurate.</td>
<td>Mathematical equations might be missing or inaccurate. Student does not include graphs to show how s/he maximized volume or the graphs are inaccurate.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Overall presentation</th>
<th>Advanced</th>
<th>Progressing</th>
<th>Beginning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student presents information in a logical, clear format. Student is convincing, speaks clearly, and can verbalize his/her reasoning in a sophisticated manner. Student is persuasive and persuades company to change packaging.</td>
<td>Student presents information clearly to audience. Student can explain mathematical reasoning to audience, but not with the level of sophistication and persuasion as those in the advanced column.</td>
<td>Student does not present information clearly to the audience. The audience has difficulty understanding the mathematical reasoning in the presentation and why the company should change its packaging.</td>
<td></td>
</tr>
</tbody>
</table>
# Algebra 2

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Introduction to Algebra 2

The debate over the necessity of Algebra 2 is over. From college entrance to carbon dating bones, Algebra 2 is essential for students and graduates to go beyond what is considered “entry level.”

Introduction

Andrew Hacker's 2012 article in *The New York Times* sparked a bit of debate when he asked readers to think about how they would answer the question in the title of his article, “Is Algebra Necessary?” Hacker's main assertion is that the mathematics sequence that most high school students follow—from geometry to calculus—sets many students up for failure and causes them to drop out of school. He cites algebra as the “onerous stumbling block for all kinds of students” (Hacker).

Hacker does note that there are careers that utilize skills learned in algebra—from creating animated movies to understanding climate change—but he argues that in the decade ahead, only 5% of entry level workers will need to be proficient in algebra and above (Hacker). So where does that leave us as teachers of students who are apt to seek real-life applications of the content and skills that they are learning? How can we keep students engaged in the material?

When teaching the concepts and skills of Algebra 2, teachers should strive to awaken a sense of curiosity in their students to engage them with the content of the course. While Hacker might be correct when he writes that only 5% of entry level workers will need to be proficient in algebra, teachers can engage students by providing them with examples of how Algebra 2 can be used in interesting contexts. For example, teachers can pose the question to students: Have you ever wondered how scientists decide how old a dinosaur bone is? The teacher can then show students how exponential functions are used in the carbon-dating of fossils. While it is unlikely that our students will ever use Algebra 2 in this context in their everyday lives, providing students with an interesting and engaging context for the use of Algebra 2 will encourage active engagement in the class.

Furthermore, whether or not we agree with Hacker’s opinion is of less importance than the fact that Algebra 2 is a requirement of our high school students if they want to attend state colleges in Massachusetts—though reading and debating his article in a math class might make for an interesting and lively conversation! If we want to prepare our students for their academic futures and ensure that academic pathways aren’t closed off to them, we need to provide them with the content and skills that they will learn in Algebra 2. The skills that they learn in Algebra 2 are essential if our students take future courses such as Calculus.

“...teachers can engage students by providing them with examples of how Algebra 2 can be used in interesting contexts.”
Algebra 2 Course Content

The Scope and Sequence for Algebra 2 covers the Massachusetts curriculum framework and is focused on developing students’ understanding of the concept of a function and the multiple function types. This course will include a brief review of linear and quadratic functions, followed by exponential and logarithmic functions, with a progression into polynomial functions. Students will complete the course with trigonometric functions and statistics. The end result is for students to understand how to build one function from another, which will prepare them for continued study of advanced functions in precalculus and calculus.

Algebra 2 students should complete this course with a thorough understanding of linear and exponential functions. They should be able to distinguish between functions with a constant rate of change vs. functions with a constant ratio of change. The expectation is that students understand what a function is and what the domain and range are. They should also have the ability to graph functions and solve linear and quadratic equations along with two variable systems of equations.

There are three key shifts in mathematics with the new state framework: greater focus on fewer topics; linking topics and thinking across grades; and pursuing conceptual understanding, procedural skills and fluency, and application with equal intensity. While all of these shifts are important for teachers to keep in mind in every course, it is important for teachers to take note of the last shift, which focuses on rigor, in Algebra 2.

To engage in rigorous tasks, students must have “a solid conceptual understanding of procedural fluency” (“Key Shifts”). This will give students confidence when they are given a complex problem to solve. While this procedural fluency is vital to solving mathematical problems, the Common Core also points out that “students must be able to access concepts from a number of perspectives in order to see math as more than a set of mnemonics or discrete procedures” (“Key Shifts”). In other words, students should not simply memorize formulas and memorize steps to solving problems. We need to teach our students that there are many ways to approach a task and we need to encourage them to use critical thinking skills to solve difficult problems.

To engage in these difficult problems, the Algebra 2 course addresses many Essential Questions that students should think about.

- Why do mathematicians make up new kinds of numbers?
- How are formulas derived from patterns?
- Why is it important to use creativity when problem-solving in math?
- Why are complex numbers useful?
- How can we collect, organize, interpret, and display data to investigate a question?
- Why is it important to interpret data carefully?
- How are patterns of change related to the behavior of functions?
- What does an equation tell you about the graph of a function?
- What strategies can be used to solve for unknowns?
- What are the benefits of, or possible uses for, representing a function in different ways (equation, graph, table, situation)?

While many of these questions are best suited to be discussed within a certain season, some of these questions can be discussed throughout the year. Where appropriate, teachers should encourage students to rethink these essential questions as they relate to new problems and new units of study.

Embedded within all three units are the Common Core State Standards for Mathematical Practice. These standards
include asking students to make sense of problems and to persevere in solving them, to reason abstractly and quantitatively, and to model with mathematics.

**Teaching Algebra 2 in DYS Schools**

Algebra 2 can be a complicated course of study for students in DYS schools for many reasons. We are asking them to work with “imaginary” numbers, asking them to work through complicated equations that require perseverance, and asking them to grapple with complex ideas that lack definitive answers. Furthermore, because students might not see a direct connection between the units of study and their lives, they might tune out and ask teachers the dreaded question: “When will I ever use this later in my life?” It would be disingenuous to tell students that all of them will use all of the content and skills that they encounter this year in their everyday lives, but we can tell students that the complex thinking and reasoning that they are doing will help them be successful in life, and we can also provide students with engaging contexts that utilize skills of Algebra 2, such as the carbon-dating example mentioned above. In every unit of study, teachers should make sure that students are playing an active role in the classroom and should make sure that students are not asked simply to do rote memorization.

If teachers are looking for ways to make direct connections to their students’ lives, the most obvious place is in the Spring Season of Trigonometry, Statistics, and Probability. Teachers can help students make sense of how these ideas relate to their lives by providing...
students with summative and formative assessments that provide students with interesting real-world scenarios, by allowing students to read articles that discuss statistics or probability, and by bringing in professionals who use some of these skills in order to make job connections that interest the students in the classroom. Teachers can show students how statistics can be skewed to manipulate people.

Because Algebra 2 can be a difficult subject for students, using a variety of instructional methods will encourage student participation and engagement. The concepts that students are learning are difficult, so teachers will need to provide students with as many hands-on activities as possible. Teachers will need to allow students to make mistakes—this is how they will learn. We can't hover over students and correct them every time they make a mistake; instead, we need to give them room to realize their errors and to find ways to correct them. This will be difficult for some students, so teachers will need to reassure students that struggling and making mistakes is expected and encouraged.

As much as possible, teachers will also provide students with videos, games, and readings to encourage learning through a variety of mediums. Additionally, teachers will want to use technology as much as possible to give students the skills that they will need to be successful in future careers. The units provided in this guide include various activities for students to engage with that should grab their interest.

Works Cited

Reading the Algebra 2 Scope and Sequence Chart

The amount of information contained in the Scope and Sequence on the following pages may seem overwhelming at first. The best way to study it is to read across from left to right. The keys below on this page offer guidance on how to properly access the Scope and Sequence Chart on pp. 6.2.2 to 6.2.3.

The Scope and Sequence is COLOR-CODED. Each color is important, and its meaning and the main highlights in DYS pedagogy are described in the key in the LEFT column below. The RIGHT column shows the Algebra 2 topics and the seasons when they are taught during the academic year.

Topics listed with an asterisk (*) in the Scope and Sequence have exemplar units in this Guide.

### Scope and Sequence Chart Key

- **Golden row**: The GOLD header rows identify columns for Topics, Emphasized Standards, Essential Questions, Transfer Goals, and Performance Assessments for each of the seasons in mathematics.

- **Green row**: The GREEN row contains the FALL season. 

- **Blue row**: The BLUE row contains the WINTER season.

- **Red row**: The RED row contains the SPRING season.

- **Gray row**: The GRAY row across the bottom contains the eight Common Core State Standards for Mathematical Practice (SMP).
## Mathematics | Algebra 2, Chapter 6

### SCOPE AND SEQUENCE

<table>
<thead>
<tr>
<th>Topics</th>
<th>Emphasized Standards</th>
<th>Essential Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Elementary Functions</strong>&lt;br&gt;1. Expressions, Equations, and Inequalities&lt;br&gt;2. Relations and Functions&lt;br&gt;3. Quadratic Functions*&lt;br&gt;4. Polynomial Expressions with Functions</td>
<td><strong>Algebra:</strong> Interpret the structure of exponential, polynomial and rational expressions (A-SSE 1a, b). Create equations that describe numbers or relationships (A-CED 1, 2, 3).&lt;br&gt;<strong>Functions:</strong> Interpret functions that arise in applications in terms of the context. (Include polynomial, rational, square and cube root, trigonometric, and logarithmic functions.) (F-IF 1, 2, 4, 5, 6). Analyze functions using different representations (F-IF 7a, 7b, 8a, F-IF. 9). Build new functions from existing functions (F-BF 3, 4). Construct linear models and solve problems involving linear models. (F-LE 2, 5).</td>
<td>What does an equation tell you about the graph of a function? What are the benefits of or possible uses for representing a function in different ways (equation, graph, table, situation)? How do multiple functions interact on the same coordinate plane? How are formulas derived from patterns? What are uses of functions in real life?</td>
</tr>
<tr>
<td><strong>Advanced Functions</strong>&lt;br&gt;1. Complex Numbers*&lt;br&gt;2. Radical Functions&lt;br&gt;3. Exponential and Logarithmic Functions&lt;br&gt;4. Rational Functions</td>
<td><strong>Numbers and Quantity:</strong> Extend the properties of exponents to rational exponents (N-RN 1, 2). Use properties of rational and irrational numbers (N-RN 3). Perform arithmetic operations with complex numbers (N-CN 1, 2, 7).&lt;br&gt;<strong>Algebra:</strong> Understand the relationship between zeros and factors of polynomials (A-APR 2, 3). Rewrite rational expressions (A-APR 6, 7). Create equations that describe numbers or relationships (A-CED 1, 2). Solve rational and radical equations in one variable (A-REI 2, 3).&lt;br&gt;<strong>Functions:</strong> Interpret functions that arise in applications in terms of the context. (Include polynomial, rational, square and cube root, trigonometric, and logarithmic functions.) (F-IF 4, 5). Analyze functions using different representations (F-IF 7b, 7c, 7e, 8b, 9, 10). Build a function that models a relationship between two quantities (F-BF 1b). Build new functions from existing functions (F-BF 3, 4). Construct exponential models and solve problems (F-LE 2, 4, 5).</td>
<td>Why are inverse operations the basis of algebraic thought? What strategies can be used to solve for unknowns? How can patterns, relations, and functions be used as tools to best describe and help explain real-life situations? How are patterns of change related to the behavior of functions? Why do mathematicians make up new kinds of numbers? Why are complex numbers useful?</td>
</tr>
<tr>
<td><strong>Trigonometry, Statistics, and Probability</strong>&lt;br&gt;1. Trigonometric Functions and Applications&lt;br&gt;2. Trigonometric Graphs and Identities&lt;br&gt;3. Data Analysis and Statistics*&lt;br&gt;4. Probability</td>
<td><strong>Functions:</strong> Interpret functions that arise in applications in terms of the context (Include polynomial, rational, square and cube root, trigonometric, and logarithmic functions.) (F-IF 4, 5). Analyze functions using different representations (F-IF 7e, F-IF. 9, F-IF.10). Build new functions from existing functions (F-BF 3). Extend the domain of trigonometric functions using the unit circle (F-TF 1, 2, ). Model periodic phenomena with trigonometric functions (F-TF 5). Prove and apply trigonometric identities (F-TF 5).&lt;br&gt;<strong>Statistics:</strong> Summarize, represent, and interpret data on a single count or measurement variable. Use calculators, spreadsheets, and other technology as appropriate. (S-ID 1, 2, 3, 4). Understand independence and conditional probability (S-CP 1, 2, 3, 4, 5). Use the rules of probability (S-CP 6, 7, 8). Make inferences and justify conclusions from sample surveys, experiments, and observational studies (S-IC 1, 2, 3, 4, 5, 6).</td>
<td>How can we use trigonometry to solve real-life problems in surveying and structures? How can we collect, organize, interpret, and display data to investigate a question? Why is it important to interpret data carefully? Could the result be due to chance, or is something else going on? How does the choice of display play a role in interpreting the data?</td>
</tr>
</tbody>
</table>

### Common Core State Standards for Mathematical Practice (SMP):

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
### Transfer Goals

#### Application of Learning

Using technology, students will make and test conjectures about the graphs of functions and draw inferences from the results. Students will use the equations, graphs, and patterns of quadratic functions to solve real-world problems and make informed decisions.

- Students will be able to collect, organize, and model data, which will allow them to discover patterns, draw conclusions, and predict future outcomes.
- Students will use their learning of simplifying, solving, and graphing equations to analyze how these equations will act as a basis for further math topics and how they relate to the world around them.

#### Performance Assessment

**Matching Equations with Graphs**

After studying all aspects of quadratic functions and learning how quadratic expressions are represented in graphs, students will accurately match a set of quadratic expressions with the appropriate graphs and explain the factors that influenced their choices by using a critical thinking game.

**Complex Numbers for Beginners**

Using the GRASPS format, students will create a proposal for a “Complex Numbers for Beginners” book that explains the concept of complex numbers, why people need to understand them, how to perform operations with them, and to represent them graphically.

**Designing, Conducting, Presenting and Publishing a Study**

Students will design, conduct, and publish their own statistical studies based on questions of interest to them. Depending on the nature of the questions, the studies could be sample studies, observational studies, or experimental studies. No matter what the nature of study is, it is important that students consider and prove or disprove the null hypothesis: Could the results be due to chance?

Each study must include the following: research question, type of study, independent variable, dependent variable, method, statistical test and/or simulation. Students will conduct their studies, collect and sort data and use appropriate statistical technique(s) to determine whether the results are significant. Students will then organize their materials and prepare brief oral reports to the class. After receiving feedback, students will plan and develop chart-paper posters, or written reports detailing their studies, including all of the elements included in the oral reports, but with a more formal presentation, including tables and/or graphs, statistical results, and analysis.

### Common Core State Standards for Mathematical Practice (continued):

| 5. Use appropriate tools strategically. | 6. Attend to precision. | 7. Look for and make use of structure. | 8. Look for and express regularity in repeated reasoning. |
Quadratic Functions

TOPIC SEASON | Algebra 2—Elementary Functions

This unit is designed for use in long-term programs.
Sections may be adapted for short-term settings.

Unit Designers: N. Koch, B. Penniman

Introduction

What do flight paths, springboard dives, and maximizing sales revenue have in common? If you’re thinking they have something to do with math, you’re right, but more specifically, we can analyze them with quadratic equations. Oftentimes, as mathematical concepts get more advanced, our students do not see the real-life applications of the concepts and skills that they learn; math simply becomes numbers and equations on a page. This unit on quadratic functions aims to change that by engaging students in real-life problems that can be solved by applying the skills and content learned here. We want our students to see that there are many applicable uses of the math that they are studying, including applications to career fields that they may enter. Besides equipping students with advanced mathematical skills that are necessary to be successful in many technical fields, this unit focuses on developing mathematical concepts and skills through inductive reasoning, which will serve our students well in other domains.

The “Quadratic Functions” unit is designed to come third in the Fall Season of Elementary Functions and will take three weeks to complete. Teachers in short-term settings have the option of breaking this long-term unit into two mini units, with Lessons 1-4 constituting a six-day sequence that introduces students to quadratic expressions and functions. Lessons 5-9 could be a separate five-day unit that focuses on quadratic growth patterns, completing the square, and the quadratic formula. The Formative Assessments throughout the current unit could be used as Summative Assessments for the mini-units.

The “Quadratic Functions” unit focuses student attention on these standards:

F-BF.3: Identify the effect on the graph of replacing \( f(x) \) by \( f(x) + k \), \( kf(x) \), \( f(kx) \), and \( f(x + k) \) for specific values of \( k \) (both positive and negative); find the value of \( k \) given the graphs. Include simple rational, radical, logarithmic, and trigonometric functions. Utilize technology to experiment with cases and illustrate an explanation of the effects on the graph. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

F-IF.2: Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. For example, given a function representing a car loan, determine the balance of the loan at different points in time.
In the unit that follows, lessons do not include attention to \( f(kx) \) or to even and odd functions. In order to help students engage with these standards, teachers should focus class discussion and student thinking around the Essential Questions of the unit.

What does an equation tell you about the graph of a function?

What are the benefits of, or possible uses for, representing a function in different ways (equation, graph, table, situation)?

When discussing these Essential Questions, teachers should encourage students to rethink and revise their answers to these questions as they engage with new ideas.

To be ready to address the standards in this unit, students should be familiar with graphing on the coordinate plane and using function notation. Function notation can be difficult for students to grasp, so teachers should use their discretion when deciding how much reinforcement and reteaching is necessary before asking students to grapple with the more complicated ideas in this unit. While the “Quadratic Functions” unit includes many challenging problems for students to solve, plenty of support has been provided to ensure that students can be successful. The extensive use of algebra tiles allows students to approach problems concretely. For example, when completing the square in Lesson 7, students will literally make a square with tiles. This will ensure that students who have trouble with abstraction are still able to complete these problems and work with these concepts.

For adaptation ideas for this unit, see p. 6.3.3 on the right and below

ADAPTATION NOTES:

For use of this unit in short-term programs (Plan 2), the teacher should re-conceptualize it as two mini-units. Add one day to each mini-unit for a Summative Assessment.

Lessons 1-4 constitute a six-day sequence that introduces students to quadratic expressions and functions and working with graphs, factoring, vertex form, and online graphing tools. Throughout this sequence, the teacher can work with students to create detailed anchor charts for each lesson that will serve not only as review tools for the students who made them but also as catch-up tools for students entering the program mid-unit. Further, the Formative Assessments included in these early lessons and portions of the culminating Performance Task can be adapted to serve as a Summative Assessment for students who leave the program before the end of the unit.

Similarly, Lessons 5-9 constitute a five-day sequence focusing on quadratic growth patterns, completing the square, and the quadratic formula. Students entering the program at this stage can review the anchor charts created in Lessons 1-4 and practice skills such as factoring and using vertex form as they learn this new material. Again, the teacher should work with students to create new anchor charts for Lessons 5-9 and may use the Formative Assessments in these lessons as well as portions of the culminating Performance Task as a Summative Assessment.
## Elementary Functions: Quadratic Functions

Adapting This Long-Term Unit for Short-Term Programs

### Plan 1 (Long)

<table>
<thead>
<tr>
<th>FALL SEASON—Quadratic Functions: Long-Term Programs</th>
</tr>
</thead>
<tbody>
<tr>
<td>MONDAY</td>
</tr>
<tr>
<td>Week 1</td>
</tr>
<tr>
<td>Lesson 1: The Function of Quadratic Functions</td>
</tr>
<tr>
<td>Week 2</td>
</tr>
<tr>
<td>Lesson 4: Investigating Parameters of Quadratics with Vertex Form</td>
</tr>
<tr>
<td>Week 3</td>
</tr>
<tr>
<td>Lesson 8: Standard Form to Vertex Form—Part 2, Continued</td>
</tr>
</tbody>
</table>

### Plan 2 (Short)

<table>
<thead>
<tr>
<th>FALL SEASON—Quadratic Functions: Short-Term Programs (divided into two mini-units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MONDAY</td>
</tr>
<tr>
<td>Mini-unit 1 Week 1</td>
</tr>
<tr>
<td>Lesson 1: The Function of Quadratic Functions</td>
</tr>
<tr>
<td>Mini-unit 1 Week 2</td>
</tr>
<tr>
<td>Lesson 4: Investigating Parameters of Quadratics with Vertex Form</td>
</tr>
<tr>
<td>Mini-unit 2 Week 1</td>
</tr>
<tr>
<td>Lesson 5: Quadratic Growth Patterns—Part 1</td>
</tr>
<tr>
<td>Mini-unit 2 Week 2</td>
</tr>
<tr>
<td>Lesson 9: The Quadratic Formula</td>
</tr>
</tbody>
</table>

See Adaptation Notes on the preceding page (p. 6.3.2)
**Emphasized Standards (High School Level)**

**FUNCTIONS**

**F-BF.3:** Identify the effect on the graph of replacing \( f(x) \) by \( f(x) + k \), \( kf(x) \), \( f(kx) \), and \( f(x) + k \) for specific values of \( k \) (both positive and negative); find the value of \( k \) given the graphs. Include simple rational, radical, logarithmic, and trigonometric functions. Utilize technology to experiment with cases and illustrate an explanation of the effects on the graph. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

**F-IF.2:** Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. For example, given a function representing a car loan, determine the balance of the loan at different points in time.

**Essential Questions (Open-ended questions that lead to deeper thinking and understanding)**

What does an equation tell you about the graph of a function?

What are the benefits of, or possible uses for, representing a function in different ways (equation, graph, table, situation)?

**Transfer Goals (How will students apply their learning to other content and contexts?)**

Using technology, students will make and test conjectures about the graphs of functions and draw inferences from the results.

Students will use the equations, graphs, and patterns of quadratic functions to solve real-world problems and make informed decisions.
### Learning and Language Objectives

By the end of the unit:

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<table>
<thead>
<tr>
<th>Students should know...</th>
<th>understand...</th>
<th>and be able to...</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Quadratic</strong></td>
<td>A quadratic expression includes a variable raised to the second power, multiplied by a non-zero coefficient. It may also include a linear term and a constant term. In the data table for a quadratic function, the first-order differences form an arithmetic sequence.</td>
<td>Identify expressions as quadratic and recognize what types of situations tend to be modeled by quadratic functions. Use the pattern in a table of values to determine if a quadratic function can fit the data.</td>
</tr>
<tr>
<td><strong>Parameter</strong></td>
<td>The same function can be represented with several different equivalent expressions. Different forms give different pieces of information about the graph of the function.</td>
<td>Identify which form is which and convert from one form to another. Match an equation to a graph and decide which form is most useful in a given situation.</td>
</tr>
<tr>
<td><strong>Root of an equation</strong></td>
<td>A root is a solution to an equation of the form $f(x) = 0$. In the domain of real numbers, quadratic equations can have one root, two roots, or no roots at all, depending on the value of the discriminant. On a graph, the roots correspond to the x-intercept(s) of the function, the place(s) where $f(x) = 0$.</td>
<td>Use the discriminant to determine how many real roots a quadratic equation has and use the graph of a quadratic function to see how many zeros it has by visual inspection. Find the roots of a quadratic equation in any form.</td>
</tr>
<tr>
<td><strong>Irrational number</strong></td>
<td>Irrational numbers were invented to solve a closure issue (the operation of square rooting on the set of rational numbers). Irrational numbers are still real numbers, i.e., they represent a distance, and they occupy a definite spot on the number line.</td>
<td>Approximate the value of an irrational number, such as the square root of 80 is close to 9 because $9^2$ is 81. Use irrational numbers to solve quadratic equations.</td>
</tr>
</tbody>
</table>
Assessment Evidence
Quality questions raised and tasks designed to meet the needs of all learners

Performance Task and Summative Assessment (see pp. 6.4.29-6.4.33)

**Aligning with Massachusetts standards**

After studying all aspects of quadratic functions and learning how quadratic expressions are represented in graphs, students will accurately match a set of quadratic expressions with the appropriate graphs and explain the factors that influenced their choices.

*Note:* This task is actually a critical thinking game, as it requires mathematical reasoning based on deep understanding of quadratic functions.

**Pre-Assessments** (see pp. 6.4.10-6.4.12)

*Discovering student prior knowledge and experience*

Lesson 1: While the instruction to the following content may seem accelerated, the teacher will assess student readiness and scaffold appropriately based on students’ skill levels. Based on knowledge from the unit on evaluating expressions, students will evaluate quadratic expressions using function notation. They will also explain what is easy to do and what is hard to do when working with functions (e.g., evaluating vs. solving).

**Formative Assessments** (see pp. 6.4.12-6.4.29)

*Monitoring student progress through the unit*

Lesson 2: Students will solve a quadratic equation about a diver by graphing and use the graph to answer specific questions about the diver’s path.

Lesson 3: Students will make a poster based on algebra tiles to demonstrate why \((x + 3)^2\) is not equivalent to \(x^2 + 9\) and explain their reasoning process to the teacher or the class.

Lesson 4: Students will use graphing software to create quadratic functions that satisfy specific conditions and explain orally or in writing how they picked the parameter values to create their functions.

Lesson 5: For each table provided, students should record the differences. They should observe where the vertex of the graph is and also what the \(a\) value is for the quadratic function. They will then use those observations to write the equation of the function in vertex form.

Lesson 6: For each equation provided, students will make a graph by reading the equation: first, students mark the vertex at \((h, k)\); then, they use slope triangles (based on the value of \(a\)) to find points on either side of the vertex.
Lesson 7: Students will convert standard form quadratic expressions into vertex form by completing the square and state whether or not they believe that any quadratic expression can be converted to vertex form, explaining their reasoning.

Lesson 8: Students will use completing the square to solve a revenue maximization problem. To demonstrate their understanding of the problem-solving process, students will also make up their own maximization problem situation. They should prepare a video or PowerPoint presentation that includes:

- a description of the situation
- an equation that models the situation, with an explanation of how they arrived at each part of that equation
- steps for finding the maximum of the function
- a solution stated in the context of the problem

Lesson 9: Students will use the quadratic formula to solve equations, estimate the value of irrational solutions, and use graphing software or a graphing calculator to check their answers.
Multiple Means of Engagement

This is the *why* of learning, what makes students engage or disengage. Throughout the unit plan, the student will be provided with as many choices in the level of challenge and complexity as possible in order to recruit and sustain engagement. For example, the teacher will encourage and support students in setting their own personal, academic, and behavioral goals. The teacher will use many strategies to guide students, including reminders, guides, rubrics, checklists, and prompts among other things that focus students on self-regulatory goals. Student tasks will be varied, allowing for active participation, exploration, and experimentation. The teacher will provide differentiated models, scaffolds, and feedback, as well as content material that is culturally relevant and responsive to student’s backgrounds. Most important is that teachers design assignments and tasks with authentic outcomes, and that are purposeful and convey meaning to real audiences.

While quadratic equations might seem too abstract to engage student interest, the unit includes real-world applications of these concepts, ranging from experiments in weightlessness to maximization of profit. More important, the unit employs an inquiry model that draws on students’ natural curiosity and powers of inference to develop and test hypotheses. In the unit opener, the students are engaged in watching a short music video to introduce them to the concept of quadratic equations. In Lessons 5 and 6, the students have an opportunity to be engaged in class discussions. Throughout the unit, there are choices for the students to make, for example the type of tools they can use to solve problems or, in Lesson 7, even choosing the format they want to express their answer. Lastly, when students have the opportunity to write questions for their peers, students become engaged and interested in how to complete the task from Lesson 8.

Multiple Means of Representation

This is the *what* of learning. There are many pathways to conveying information to students. Throughout the unit, the teacher will provide information and materials in several modalities such as diagrams, vocabulary cards, and word walls, posters, and charts with formulas for calculations; and models, videos, and audio for text. The teacher will also demonstrate concepts through hands-on activities, especially the opportunity for students to use algebra tiles starting in Lesson 3.

This unit places particular emphasis on the relationships between mathematical equations and expressions and the graphs that are used to model them. Students will learn to “read” one to visualize the other. Avoiding textbooks entirely, this unit relies on a variety of “texts” to represent key concepts, many of them generated by students themselves on paper and online, however, if the teacher decides that students will benefit from the use of a textbook or graphs, the teacher may provide these tools. The teacher will be utilizing the Desmos Graphing Calculator online resource starting in Lesson 4 and going through the rest of the unit to demonstrate concepts. In the unit, the teacher will be presenting videos to the class to provide examples of mathematical connections, especially highlighting real-world connections, in Lessons 1, 2, and 7. In addition, in Lesson 7 the teacher will provide a PowerPoint to instruct the students in the class.
Multiple Means of Action and Expression

This is the how of learning. In learning activities students will be provided options for demonstrating what they know and can do. Students will have access to assistive technology and use multiple media. For example, students will have access to word processors with grammar checks, word prediction, and spell checkers. Students could complete projects by making PowerPoint presentations found in Lesson 7, rapping through music videos, or performance, or drawing illustrations. In addition, students will have access to calculators. The teacher will scaffold writing or composing activities using tools such as concept maps, outlining tools, or graphic organizers. Students may need sentence starters and story webs to complete writing or composing tasks. The teacher will also break down long-term goals into short-term reachable goals.

Students will have a variety of opportunities to express what they have learned and are learning in this unit, including informal writing and graphing (on paper and online), arithmetic and algebraic computation, and especially talk: conversations among peers, especially during Lessons 2, 5, and 6; Socratic interactions with the teacher; and occasional formal presentations. During Lesson 1, students are given the choice to represent their answers through writing or drawing; moreover, in Lesson 3, students can demonstrate their learning by posters using algebra tiles. Lesson 7 encourages students to learn to use graphing software to complete the work. The final product permits students to decide what mathematical method would work the best to complete the final Performance Task.

For Empower Your Future Connections, see p. 6.5.1
Literacy and Numeracy
Across Content Areas

Reading
Students will read internet articles, take notes, and discuss concepts with peers. They will also “read” and interpret algebraic expressions and equations and their corresponding graphs.

Writing
Students will write guided notes, Exit Tickets, and other writing-to-learn activities explaining their understandings of key concepts and speculating about problems not discussed in class through inference and analysis.

Speaking and Listening
Students will contribute to class discussions of key concepts by making inferences from class presentations, stating their hypotheses, and explaining their reasoning. They will also make formal presentations of their work to peers.

Language
Students will learn and use appropriately vocabulary related to quadratic expressions and equations.

Numeracy
Students will employ arithmetic and algebraic operations when working with quadratics. They will also use inference and interpretation when creating and interpreting graphs.

Resources (in order of appearance by type)

Websites
Lesson 1

“NASA.” NASA. National Aeronautics and Space Administration. 2017. www.nasa.gov


Lesson 2

“MIT Physics Demo Strobe of a Falling Ball.” YouTube. mittechtv. 2009. www.youtube.com/watch?v=x4znShlK5A


Lessons 4, 6, 7

Lesson 7


Lesson 8

“Changing a Quadratic from Standard Form to Vertex Form.” YouTube. MrsQuesnelle. 2009. www.youtube.com/watch?v=dse3XdNq92k


Lesson 9
“Proving the Quadratic Formula.” YouTube. ThinkWellVids. 2014. www.youtube.com/watch?v=zThTk2FbdLY


Lesson 10


Materials
Chart paper, algebra tiles

Lesson 1: Weightless Video Spreadsheet
Lesson 4: Investigating Quadratics in Vertex Form
Activity Sheet pp. 6.6.1-6.6.10
Lesson 10: Quadratic Functions Practice Task
(Activity Sheet pp. 6.6.11-6.6.13)
Lesson 10: Quadratic Functions Performance Task
(Activity Sheet pp. 6.6.14-6.6.19)
Lesson 10: ANSWER KEY—Quadratic Functions
Answer Key p. 6.6.20

PREREQUISITES: Math skills needed for this unit

*Quadratic Functions* is the third unit in the Fall Season of Algebra 2. The prerequisite math skills summarized below are usually taught in the units that precede *Quadratic Functions* (see Scope and Sequence chart). The following skills will be needed for students to successfully complete this unit.

**Students should know:**

- How to use function notation and graphing on the coordinate plane
- How to evaluate and simplify quadratic expressions and to identify equivalent expressions
- The difference between an expression and an equation
- How to use inverse operations to solve an equation
- How to use algebra tiles or a concrete model to display an expression
- Key vocabulary: *term, coefficient, exponent*

**Outline of Lessons**

Introductory, Instructional, and Culminating tasks and activities to support achievement of learning objectives
INTRODUCTORY LESSON

Stimulate interest, assess prior knowledge, connect to new information

Lesson 1

The Function of Quadratic Functions

Goal
Students will recognize and demonstrate that quadratic functions are used to model and solve real-world problems.

Do Now (time: 5 minutes)
Students will respond to the following prompt in writing and/or drawing:

- How do astronauts practice being weightless before they go into space?
- Is there any way to experience weightlessness on Earth?

Students should compare their responses to the prompt.

Hook (time: 5 minutes)
Students will watch the OK Go “Upside Down and Inside Out” music video at the website below.
The teacher should start the video at :10, after the explanation that it was filmed on a plane in the sky. After watching the video, students should reconsider their responses to the Do Now prompt by completing a turn and talk with their peers, and if no one guesses, the teacher can explain that the video was made in an aircraft accelerating downward.

**SEE:** Upside Down and Inside Out
http://okgo.net/2016/02/11/upside-down-inside-out

Note: This hook is engaging for youth, but the teacher may also want to include at some point during the unit other more diverse representations of individuals experiencing zero gravity by researching the NASA website.

**SEE:** NASA
www.nasa.gov

Presentation (time: 20 minutes)
The teacher will briefly introduce the unit, explaining that students will be extending their learning about quadratic functions, how they are graphed and how they are used. The teacher may wish to give a few illustrations, such as \( ax^2 + bx + c \) as the standard form of a quadratic expression. By definition, a quadratic expression can be written in the form \( ax^2 + bx + c \) where \( a \) does not equal zero.

The teacher will explain that the recording of the OK Go video involved solving a quadratic equation. The teacher will project the article “The Physics of OK Go’s Epic New Zero-G Video” and distribute hard copies to students. The teacher will then read the first part of the article aloud as students follow along, pausing to clarify and discuss the examples (without delving too deeply into the physics of the situation). The teacher will then highlight the problem:

*Could you record this video in one take?*
If you look at the whole music video, the weightlessness part lasts about 166 seconds. Could you fly an aircraft to produce weightlessness for this long? The teacher will lead students through the remainder of the article, highlighting the dynamic graphs, which show height as a function of time.

SEE: The Physics of OK Go’s Epic New Zero-G Video
www.wired.com/2016/02/the-physics-of-ok-gos-epic-new-zero-g-video

Practice and Application (time: 25 minutes)
The teacher should then introduce the task for the lesson, which involves using the quadratic functions that produced the graphs in the article.

In the first scenario, with an initial altitude of 10,000 feet and an initial velocity of 500 miles per hour, the equation for height as a function of time is:

\[ h(t) = -16t^2 + 518.45t + 10,000 \]

Note: The linear coefficient of 518.45 represents the vertical component of the initial velocity, expressed in feet per second (500 miles per hour is 733 feet per second; this is multiplied by the sine of the initial angle to capture just the vertical part of the velocity). It might be appropriate to discuss the derivation of that coefficient as an extension, but NOT in the initial presentation.

At this point, the students should practice evaluating the expression and using function notation, e.g.,

What is \( h(3) \)? What does the value of \( t \) need to be for \( h(t) \) to equal 12,000 feet? Why is there more than one answer to that question? They should be able to tie the function notation to the context and understand that \( h(3) \) means the height of the plane at a time of 3 seconds. To maintain a connection to the context, encourage students to use units in their answers.

The teacher should emphasize that the issue for the band is having enough time to film the video. So they need to know how much time they will have before the plane comes back down to a certain altitude. That is why they are looking for the value of \( t \). If students plugged the desired length of time into the equation, \( h(166) \), the result would be a negative height, meaning, of course, that the plane crashed.

The students are not expected to know how to solve a quadratic equation, and the teacher should not attempt to use formal methods to do so. With a spreadsheet or a graphing calculator’s table capability, students can simply inspect the table of values to look for things like when is \( h(t) = 10,000 \text{ feet} \)? And of course it won’t be exactly 10,000 feet, so this will allow discussion of which value is closest. It’s a good time to learn how to use the “delta table” setting of the graphing calculator, so that with smaller increments in the value of \( t \), students can get closer and closer to the desired output value. This process of successive approximation can be very effective, while it also motivates the formal solving techniques that students will use later in the unit.

In the second scenario, with an initial altitude of 30,000 feet and an initial velocity of 600 miles per hour, the equation for height as a function of time is:

\[ h(t) = -16t^2 + 622.254t + 30,000 \]

When students look for the value of \( t \) such that \( h(t) = 15,000 \text{ feet} \), they should find that it’s somewhere around 55 seconds as stated in the article. This is still not enough time to film the video, so students might want to change the initial conditions again, perhaps to a faster speed or a higher altitude. They can
make these changes with the spreadsheet WeightlessVideo.xlsx, available in the Math Guide resources on Google Drive.

SEE: Weightless Video Spreadsheet

Review and Assessment (time: 5 minutes)
Next, students can turn and talk or the teacher may discuss their reflections as a whole group. Students should explain or write what was easy to do and what was hard to do in working with the functions. For example, was it easier to start with a specific time and find the height or start with a desired height and find the time? The teacher could point out that this is the difference between evaluating a quadratic expression and solving a quadratic equation and discuss why solving is harder. Students might mention that the guess-and-check aspect takes too much time or that they would like to have a more direct method of finding the number. At this point, the teacher can mention that there are many equations in mathematics which cannot be solved by a direct method; guess and check is the only option. However, it turns out that quadratic equations can be solved directly, and students will be learning techniques to do this during the unit.

Extension
For further investigation of the physics of weightlessness, students can visit the “Why Do Astronauts Float around in Space” link in the article below and/or “Feeling ‘Weightless’ When You Go Over the Hump.”

SEE: Why Do Astronauts float Around in Space?
Feeling ‘Weightless’ When You Go Over the Hump
http://hyperphysics.phy-astr.gsu.edu/hbase/mechanics/hump.html

INSTRUCTIONAL LESSONS
Build upon background knowledge, make meaning of content, incorporate ongoing Formative Assessments

Lesson 2
Solving Problems with Graphs

Goal
Students will explain the connection between a graph and a situation and demonstrate how to solve a problem graphically.

Do Now (time: 5 minutes)
Students will review the illustration that follows on p. 6.4.13 showing a graph of the OK Go flight path discussed in Lesson 1 and list five pieces of relevant information about the flight that they can read or interpret from the graph (e.g., starting and ending altitude, length of flight, maximum altitude, time at maximum altitude). Students should compare their answers.
Hook (time: 5 minutes)
After explaining that this lesson will focus on a similar problem, but based on sports, the teacher will ask the students to give examples of sports motions that could be modeled by quadratic equations (a jump shot in basketball, the flight of a javelin in track and field, a field goal in football, a dismount in gymnastics, etc.). Then the teacher will show a portion of the video below of dives from a 3-meter board and ask students to predict how the graph of one of these dives would be similar to and different from the flight graph from Lesson 1.

SEE: 2008 NCAA Men’s 3 Meter Diving Final
www.youtube.com/watch?v=2IOHxg8wJbg

Note: Students will probably realize that the graph will go lower than the starting point because the diver hits the water. They may assume, incorrectly, that the curve will be steeper because the diver goes almost straight up and down. With graphs of height vs. time, it is a very common misconception to see the graph as a direct picture of the object moving through the air.

Presentation (time: 15 minutes)
It’s important for students to realize that the quadratic function is measuring height vs. time. Horizontal motion is not part of the equation, and the graph is not a picture of the event. To show why it’s believable that the height function is quadratic, the teacher should show this video that uses strobe photography:

SEE: MIT Physics Demo Strobe of a Falling Ball
www.youtube.com/watch?v=xQ4znShlK5A

In strobe photography, snapshots are taken at equal time intervals. This allows the viewer to see height as a function of time. The teacher should freeze the final image so that students see how the height changes with each time step.

The photo, “The Projectile Drop” by Loren M. Winters (website follows on p. 6.4.14) was also taken with strobe photography. It shows that the effect of gravity is the same whether something drops straight down or has horizontal motion. Both of these objects would have the same graph of height vs. time, because at each time interval, they are at the same height above the ground. The teacher should ask students why
that graph is not linear. Students should be able to articulate that there is an increasing amount of space between each snapshot of the ball. If this were a linear situation, the yellow lines would be equally spaced. Some students may see the red ball as linear, because it falls straight down. To counter this misconception, the class should use chart paper at the front of the room to make a graph of height vs. time. One student can use a meter stick to measure the height and then call out the measurement to another student who plots that height measurement on the graph. (To save time, the teacher can set up the axes on the chart paper before class so that it is ready for students to use.)

**SEE:** The Projectile Drop
http://courses.ncssm.edu/aphys/photos/projectile_drop.htm

The teacher will then introduce the diving function that the lesson will focus on (adapted from Illustrative Mathematics F-IF; A-REI Springboard Dive).

**SEE:** Springboard Dive
www.illustrativemathematics.org/content-standards/tasks/375

The teacher will encourage students to consider the function and talk a little about the situation.

Suppose \( h(t) = -5t^2 + 10t + 3 \) is the height of a diver above the water (in meters), \( t \) seconds after the diver leaves the springboard.

Students may notice it is in meters whereas the airplane problem was in feet. They may not realize how tall 3 meters is. There may also be some confusion about where the height is being measured on the diver. A diver could be close to 2 meters tall, so this actually matters.

Are we measuring to the diver’s feet? The diver’s head? Or is it whatever part of the diver is closest to the water?

It is also worth discussing why the expression is a mixture of positive and negative coefficients.

What factors would increase the diver’s height above the water, and what would decrease the height?

**Practice and Application** (time: 20 minutes)

Students will make a neat and clear paper-and-pencil graph of the function. It is very important that they choose a good scale, and they may end up changing their minds about that a few times when they first start drawing the graph. The teacher may also scaffold instruction by providing graphs that include partial solutions to prompt students in arriving at the desired outcome. Typically students will choose to mark the \( t \)-axis using 1 block to represent 1 second, but that won’t work out well in this situation. To be able to demonstrate the situation effectively with their graph, they will need to stretch out the scale on the horizontal axis, since most of the action takes place between 0 and 2. When they have made a table of values, they may see that most of the values are negative. They should discuss why, in the context of the situation, we are interested only in positive values of \( h \).

Students will use the graph as a basis for answering questions about the diver:

How high above the water is the springboard? Explain how you know.

When does the diver hit the water?

At what time on the diver’s descent toward the water is the diver again at the same height as the springboard?

When does the diver reach the peak of the dive?
Once students realize that the height of the springboard is 3 meters—because it’s \( h(0) \)—they should be able to mark on the graph where that is. They should also mark where the water is and realize that the diver hits the water when \( h(t) = 0 \), or when the curve intersects the \( t \)-axis. They can estimate the value of \( t \) from the graph. When drawing the graph, they will need to decide where the vertex is. This equation happens to work out nicely so that the vertex occurs at \( t = 1 \). Students will be able to see this from symmetry in the table: \( h(0) = 3 \) and \( h(2) = 3 \).

**Review and Assessment** (time: 10 minutes)

Students will compare their labeled graphs with a partner. The students will explain how they determined the answers to the four questions. Then, after viewing a portion of the diving video again, students will evaluate how realistic they think the equation is. (Possible prompts to spur thinking: If the diver starts out at a height of 3 meters, do we believe that \( h(1) = 8 \)? What would that be saying? Do we think it would take exactly 1 second to reach the peak? Do we believe the peak is 5 meters above the height of the board?) They should also identify the part of the equation that would need to be changed to make the equation more realistic (10\( t \)). This will entail a discussion of the coefficient on the quadratic term—why it’s always the same due to the effect of gravity.

As time allows, the teacher may extend the discussion by asking students to consider how a quadratic equation modeling a motion in a different sport would be similar to and differ from the diving equation. For example, in a jump shot, gravity is the same, but for most players the point of release is lower than the goal. During the discussion, the teacher should reinforce the idea that they are measuring height vs. time. The shape of the graph is not a picture of the event. The graph doesn’t capture any horizontal motion. It’s just height at a given point in time.

**Extension**

Students could use the video of NCAA divers and a stopwatch to gather some real data about the diver’s height vs. time and try coming up with a more realistic equation. The issue of the diver flipping around should be rich for discussion. A lot of students will want to make the graph do little circles as it goes down. The teacher may utilize robotics or littlebits.cc materials to create an arc through programming, movement, or designing of a gadget to produce an arc on paper.

**Lesson 3**

Solving Equations with Factoring (2 Days)

**Goal**

Students will represent quadratic expressions concretely with algebra tiles, seeing equivalence, and factoring quadratics as a way to find exact solutions for quadratic equations.

**Do Now** (time: 5 minutes)

The teacher should ensure that students are familiar with algebra tiles. If students are not, the teacher will explain and illustrate that algebra tiles are mathematical manipulatives that allow students to better understand ways of algebraic thinking and the concepts of algebra.
Using algebra tiles, students will multiply the following expressions and represent them as rectangles:

\[(x + 2)(x + 3)\] \[\quad(x − 1)(x + 4)\]

**Hook** (time: 10 minutes)
The teacher should introduce the lesson as follows:

In the previous lesson, we used the graph of a quadratic function to estimate the time when the diver hit the water. But how do we find solutions to quadratic equations without estimating from a graph or a using a calculator? Can we use the same procedures that we used for solving linear equations?

Students may suggest the same kind of “undoing” procedure that they learned for solving linear equations. Given a quadratic in standard form with three terms, they will hit a point where undoing does not work. So then what? The teacher should ask, How do we solve those?

**Presentation** (time: 20 minutes)
With the teacher’s guidance, students will look at several quadratics in different forms to see if they can find roots for any of them. Students should see that some will be easy and some will be hard. For example:

Suppose \(f(x) = (x + 3)(x − 2)\); when does \(f(x) = 0\)?:

Suppose \(g(x) = x^2 + 4x − 5\); when does \(g(x) = 0\)?:

Suppose \(g(x) = x^2 + 4x + 5\); when does \(g(x) = 0\)?:

Suppose \(h(x) = 3(x + 3)(x − 2)\); when does \(h(x) = 0\)?:

Suppose \(j(x) = 2(x − 3)^2 − 8\); when does \(j(x) = 0\)?:

The examples in factored form may raise the zero product property, which will make the quadratics seem much easier to solve. Since standard form is hard to solve, and factored form is easy to solve, the teacher should discuss the possibility of going from standard form to factored form—that’s the goal of today’s activity—and figuring out when that can be done.

The teacher should emphasize that the product of two binomials is a multiplication problem. Multiplication can be seen by looking at the area of rectangles, e.g., \(4 \times 3\) is 4 rows of 3. This concrete approach should have been developed during Unit 1 of this course, Expressions and Equations.

**Practice and Application** (time: 45 minutes)
This activity will be divided over two days. The teacher should review the first day’s work at the beginning of the second day.

Students will use algebra tiles to represent a variety of quadratic expressions such as the following:

\[x^2 + 4x + 5\] \[x^2 + 6x + 5\] \[x^2 + 4x − 5\]

\[x^2 + 6x + 12\] \[2x^2 + 10x + 12\] \[x^2 − 25\]

Students should notice which expressions can be arranged into a rectangle and which cannot. When the constant term is not a prime number, students will need to be sure they proceed systematically and check each factorization of the constant to see if it makes the linear term come out correctly. The teacher should show them how to write the same expression in two different ways (standard form and factored form), including some expressions in which the quadratic coefficient is not 1.
Review and Assessment (time: 30 minutes)
Students will make a poster based on algebra tiles to demonstrate why \((x + 3)^2\) is not equivalent to \(x^2 + 9\) and explain their reasoning process to the teacher or the class.

Extension
Students can use a spinner or some other type of random number generator to get integer coefficients for a quadratic expression in standard form then use algebra tiles to try to factor the expression. They should observe that the vast majority of quadratic expressions cannot be factored.

Lesson 4
Investigating Parameters of Quadratics with Vertex Form (2 Days)

Goal
Students will explore the role of parameters \(A\), \(H\) and \(K\) for a quadratic written in vertex form. After experimenting with these parameters, they will reach their own conclusions about their roles in quadratic functions.

Do Now (time: 10 minutes)
Given a pair of quadratic expressions,\
\((x – 4)(x + 1)\) and \(x^2 – 5x – 4\),
students will use algebra tiles to decide if the two expressions are equivalent or not.

Hook (time: 5 minutes)
The teacher may remind students that one way to check if two expressions are equivalent is to graph them as functions. Using Desmos Graphing Calculator, the teacher will project graphs of \(f(x) = (x – 4)(x + 1)\) along with \(g(x) = x^2 – 5x – 4\). Students will see that the two graphs are different, thus the expressions are not equivalent. The teacher should ask students how we could change the expression for \(g(x)\) to make the two graphs the same.

SEE: Desmos Graphing Calculator
www.desmos.com/calculator

Note: When using Desmos, it will help to use “Projector Mode” to make the graphs thicker and more visible. This setting is available from the Settings menu (look for the wrench icon) found on the right side of Desmos. It’s also helpful to change the “Step” setting to 1.

Presentation (time: 15 minutes)
With the graphs still showing on the screen, the teacher should ask students to identify key features of the graph. They should be able to use the vocabulary of “x-intercept” and “y-intercept” and also the relevant function notation such as \(g(0)\) to talk about the y-intercept. The teacher should introduce the term “zero” as a synonym for x-intercept. The zero of a function is the value of x such that \(f(x) = 0\). It’s the input that forces the output to be zero. A word wall or word splash will be helpful for all students to support the acquisition of academic language.
They may also want to talk about the graph’s minimum point as a key feature. The teacher should introduce the term “vertex” which refers to either the minimum or maximum point of a parabola. Use Desmos to hover over the gray circle at the vertex. They will see that the coordinates are 1.5, -6.25.

The teacher will ask students to look at the expressions for \( f(x) \) and \( g(x) \). Would it be possible to read the expressions and know where the zeros are, even before we draw the graph? What about the \( y \)-intercept—how would we know that it is going to be at -4? And what about the vertex? Is there anything in the expressions to tell us where that is going to occur? Students will realize that the numbers 1.5 and -6.25 don’t appear anywhere in either of the two expressions. The teacher should continue the presentation as follows:

So far we have seen quadratic expressions written in two different forms:

**Standard Form:** \( f(x) = 3x^2 + 5x - 10 \)

**Factored Form:** \( g(x) = (x - 4)(x + 7) \)

There is another way to write a quadratic, known as vertex form. It looks like this:

\[
 h(x) = -2(x - 5)^2 + 6
\]

Today and tomorrow you will use Desmos to experiment with quadratic functions written in vertex form. Your goal is to see what you can figure out about the graph just from reading the equation.

At this point, the teacher can use the projection to point out key features of using Desmos, so that students will be familiar with its use in the following activity.

**Practice and Application** (time: 70 minutes spread over 2 days)

Students will complete the “Investigating Quadratics in Vertex Form” Activity Sheet found on pp. 6.6.1-6.6.10 of the Supplement. The teacher will support students as needed while they complete the work, providing scaffolding where necessary but still enabling students to experiment and discover. Students will expect to try lots of things, some of which will not work. That’s okay. Even when something doesn’t work, it gives them more evidence they can use to build a conclusion.

The activity may also raise some misconceptions about numbers. For example, students may believe that there is no number between .9 and 1. Or, students may not think of zero as being a number. A very common misconception is for students to confuse additive and multiplicative relationships. They tend to think of negative numbers when they need a small number. It’s important to emphasize that “small” means close to zero. Negative numbers should be associated with the idea of “opposite.”

**Note:** Students may not complete the entire activity. The teacher can decide if spending a third day would be worthwhile or not.

**Review and Assessment** (time: 10 minutes)

Students will use graphing software to create quadratic functions that satisfy specific conditions. Each student will be given a different set of conditions to satisfy. For example, one student may be asked to make a function that:

- passes through the point (-6, 2)
- forms an upside-down “U” shape
- is very wide at its vertex
The student will explain orally or in writing how he or she picked the parameter values to create the function. The student should also demonstrate how he or she can be sure that the function really does pass through that point. That is, he or she should show by using the arithmetic of the expression that \( f(-6) = 2 \).

This exercise provides a good opportunity for differentiation in the classroom because the teacher can assign different sets of conditions with varying level of challenge. In the example above the student is asked to make a curve that passes through the point \((-6, 2)\). To make that problem more challenging, the teacher might ask a student to make a curve that passes through the point \((-6, 2)\) but NOT at the vertex. Or, the teacher might ask a student to make a curve that passes through \((-6, 2)\) AND the point \((-5, 0)\).

**Extension**

The teacher can give students several cubics written in this form: \( f(x) = A(x - H)^3 + B(x - H) + K \). Be sure to include some where \( A \) and \( B \) are opposite in sign. Have students experiment and figure out the role of each parameter.

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**Lesson 5**

**Quadratic Growth Patterns—Part 1**

**Goal**

Students will discover that quadratic functions have constant second-order differences. They will see the connection between the value of the difference and the value of the leading coefficient in the quadratic.

**Do Now (time: 5 minutes)**

Then the teacher should give students the equation \( f(x) = 2(x - 4)^2 + 1 \), and tell them that one of the three graphs on the opposite page (p. 6.4.20) goes with that equation.

The teacher will ask the students to pick the graph that correctly matches the equation (it is Graph 2).

**Hook (time: 5 minutes)**

Students should explain how they picked their matching graph. Their understanding of the role of Parameter A is probably not fully formed yet—they may be confused about whether a higher value for A makes the graph steeper or flatter. It’s good to bring that confusion to the fore. The teacher will tell students that today’s lesson will give them a clearer picture of Parameter A—how and why it does what it does.

**Presentation (time: 20 minutes)**

The teacher should present tables for \( f(x) \) and \( g(x) \) found on the opposite page (p. 6.4.20).

The teacher will ask students to use patterns they see to fill in next three rows of each table. They should be able to explain orally how they used the pattern. Then ask students if \( f(x) \) will ever catch up to \( g(x) \) and explain how they decide. These questions will probably prompt students to look at differences.
FALL QUADRATIC FUNCTIONS

Lesson 5—Do Now Activity

GRAPH 1

Lesson 5—Presentation Activity

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>x</th>
<th>g(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.5</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>2</td>
<td>36</td>
</tr>
<tr>
<td>3</td>
<td>14.5</td>
<td>3</td>
<td>37</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>4</td>
<td>114</td>
</tr>
<tr>
<td>5</td>
<td>22.5</td>
<td>5</td>
<td>153</td>
</tr>
<tr>
<td>6</td>
<td>28</td>
<td>6</td>
<td>192</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>

Permission for use granted by Desmos.
After the considering the tables for \( f(x) \) and \( g(x) \) (p. 6.4.20), students will then come to the board to fill in another table, with \( x \) values starting at 1. Each student will have a different function:

\[
\begin{align*}
    f(x) &= 1x^2 \\
    g(x) &= 2x^2 \\
    h(x) &= 3x^2
\end{align*}
\]

The teacher will ask students to find the first order differences. Establish a standard format for showing the differences, such as the example at right.

Then students will stand back and compare the tables by looking at the growth patterns: How are they similar and how are they different? If you had another function, \( m(x) = 4x^2 \), what do you think you would see when you look at the differences? Students should fill in a table for \( m(x) \) and check to see if hypotheses are correct. Some students may have the idea to look at the “differences of the differences” (also known as second-order differences). They should notice that the first-order differences increase at a constant rate.

Now the teacher should project graphs for those same functions. The graphs must include the grid. Each student will go up to the board for the function that he or she worked on before and draw slope triangles starting from the vertex. So, for \( f(x) = 1x^2 \), the triangles are over 1, up 1; then over 1, up 3; over 1 up 5 and so forth. For \( g(x) = 2x^2 \), the triangles are over 1 up 2; over 1 up 6; over 1 up 10 and so forth. Students should see that the triangles for \( 2x^2 \) are twice as tall as the triangles for \( 1x^2 \). Similarly, the triangles for \( 3x^2 \) are three times as tall.

**Practice and Application** (time: 15 minutes)

The teacher will ask:

- Why do quadratic functions grow this way?

Remind students that “squaring” really comes from physically making squares. Bring out the unit squares from the algebra tiles. Have students start with a 2 by 2 square. Then ask them what they need to make that grow into a 3 by 3 square. Continue that pattern for a while until they can predict in advance how one square will grow into the next. The teacher could use this squaring activity to illustrate to students how to compute the amount of floor covering needed for a closet using the same squares.

Students can be paired. The teacher asks students how to show the growth of \( 2x^2 \), instead of just \( 1x^2 \). They should arrive at
the idea that they need to make two separate squares each time. To get from one picture to the next, they need twice as many tiles, so the growth rate for $2x^2$ will always be twice the growth rate for $1x^2$. Students should extend this reasoning to $3x^2$, $4x^2$, and so forth. They can use sketches rather than actual algebra tiles.

Then the teacher should ask students about $ax^2$, where $a$ could be any number.

How does the growth rate for $ax^2$ compare to the growth rate for $1x^2$?

The abstraction may be hard for them to articulate, but students should see that the growth rate for $ax^2$ is $a$ times as much as the growth rate for $1x^2$, because they would need to make different squares in order to model it with algebra tiles.

**Review and Assessment** (time: 10 minutes)

For each table below, students should record the differences. They should observe where the vertex of the graph is and also what the $a$ value is for the quadratic function. They will then use those observations to write the equation of the function in vertex form.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$x$</th>
<th>$g(x)$</th>
<th>$x$</th>
<th>$h(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>8</td>
<td>1</td>
<td>-10</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>-3</td>
<td>5</td>
<td>2</td>
<td>-13</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>3</td>
<td>-10</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>5</td>
<td>4</td>
<td>-1</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>0</td>
<td>8</td>
<td>5</td>
<td>14</td>
<td>8</td>
<td>18</td>
</tr>
<tr>
<td>1</td>
<td>13</td>
<td>6</td>
<td>35</td>
<td>9</td>
<td>32</td>
</tr>
</tbody>
</table>

**Lesson 6**

**Quadratic Growth Patterns—Part 2**

**Goal**

Students will apply their understanding of quadratic growth patterns to quadratics with fractional and negative leading coefficients.

**Do Now** (time: 5 minutes)

This is the same $f(x)$ function that students used in Lesson 5. Students will use the pattern they observed previously in order to work backwards in the Do Now Activity table on the next page (p. 6.4.23). For this Do Now Activity, students should fill in $f(0)$, $f(-1)$ and $f(-2)$.
Lesson 6
Do Now Activity

Fill in the blanks:
f(0), f(-1), and f(-2):  
<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>10.5</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>14.5</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>22.5</td>
</tr>
<tr>
<td>6</td>
<td>28</td>
</tr>
</tbody>
</table>

Hook (time: 5 minutes)

Students should explain how they used their pattern to fill in the table. Most students will be able to find that \( f(0) = 10 \) fairly easily, but they may struggle with what to do next and they may come up with some unusual patterns that follow a different logic. The teacher will ask students:

Do these patterns still work with decimals? What do we do if we end up with negative numbers for our differences? This is what we will figure out today.

Presentation (time: 15 minutes)

With the table for \( f(x) \) projected at the front of the room, the teacher should fill in the values so that \( f(-1) = 10.5 \) and \( f(-2) = 12 \), following a quadratic pattern for the table. The teacher should then show students how to record the differences correctly. The rule is:

If the y value drops, the difference is recorded as a negative.

Since \( f(-2) \) is 12 while \( f(-1) \) is 10.5, the difference is -1.5. The teacher should ask students to look at the table and state where the vertex of the graph would be. They should be able to say it’s at \((0,10)\) based on the symmetry of the values in the table.

Using the Desmos Graphing Calculator, the teacher should project a graph for \( f(x) = .5x^2 + 10 \) while keeping the equation hidden (Desmos allows you to hide the list of equations with the << button). Students should then come to the board and draw in slope triangles on the projected graph. They should see that the graph’s pattern is: over 1, up \( \frac{1}{2} \); over 1, up \( \frac{3}{2} \); over 1, up \( \frac{5}{2} \); and so forth.

SEE: Desmos Graphing Calculator
www.desmos.com/calculator

The class should then discuss what the value of \( a \) is for this quadratic function. This discussion should incorporate the previous day’s observations from Lesson 5, where students saw that the triangles for \( 2x^2 \) are twice as tall as the triangles for \( 1x^2 \), or more generally, the triangles for \( ax^2 \) are \( a \) times as tall as the triangles for \( 1x^2 \). Does that observation still work when the slope triangles are shorter than those for \( 1x^2 \)?
The teacher should allow students to speculate about the value of $a$ for $f(x)$, while still keeping the equation hidden. For each speculation students offer, the teacher should enter the suggested equation into Desmos to see if it matches the graph for $f(x)$. Some students may suggest a negative number for the value of $a$, because they believe that negative numbers are “smaller” than positive numbers. This is a good misconception to bring to the fore, because the graph will flip upside down when the equation is entered.

The discussion should close with the conclusion that yesterday’s rule still holds: the triangles for $ax^2$ are $a$ times as tall as the triangles for $1x^2$. For a fraction like $\frac{1}{4}$, then the triangles will be $\frac{1}{4}$ the size of the triangles for $1x^2$. Negative $a$ values work as opposites: the slope triangles for $-1x^2$ are the same size as the triangles for $1x^2$, just reflected upside down.

**Practice and Application** (time: 15 minutes)

Students will work in groups of two or three and use the observations from the previous class. For each table below, students should record the differences. They should observe where the vertex of the graph is and also what the $a$ value is for the quadratic function. They will then use those observations to write the equation of the function in vertex form.

**Note:** For the function $j(x)$, students will need to extend the pattern in the table one more row to observe the location of the vertex.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$x$</th>
<th>$g(x)$</th>
<th>$x$</th>
<th>$h(x)$</th>
<th>$x$</th>
<th>$j(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>28</td>
<td>1</td>
<td>-9</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>-57.5</td>
</tr>
<tr>
<td>-3</td>
<td>14</td>
<td>2</td>
<td>-6</td>
<td>5</td>
<td>2.25</td>
<td>5</td>
<td>-35</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>3</td>
<td>-5</td>
<td>6</td>
<td>2</td>
<td>6</td>
<td>-17.5</td>
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<tr>
<td>-1</td>
<td>-2</td>
<td>4</td>
<td>-6</td>
<td>7</td>
<td>2.25</td>
<td>7</td>
<td>-5</td>
</tr>
<tr>
<td>0</td>
<td>-4</td>
<td>5</td>
<td>-9</td>
<td>8</td>
<td>3</td>
<td>8</td>
<td>2.5</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>6</td>
<td>-14</td>
<td>9</td>
<td>4.25</td>
<td>9</td>
<td>5</td>
</tr>
</tbody>
</table>

**Review and Assessment** (time: 15 minutes)

For each equation below, students will make a graph by reading the equation: First, they will mark the vertex at $(h, k)$. Then they will use slope triangles (based on the value of $a$) to find points on either side of the vertex.

\[
f(x) = 2(x - 3)^2 - 4 \quad g(x) = -3(x - 1)^2 + 8 \]
\[
h(x) = .5(x - 3)^2 - 2 \quad j(x) = -1.5(x - 2)^2 + 4.5
\]

**Extension**

The teacher could give students a table for a cubic function, with data for $x$ values from 1 to 7. The teacher will ask them to find a pattern in the table and use it to find $f(8), f(9)$, and $f(10)$. 

Massachusetts DYS Education Initiative—Mathematics—2017 Edition | Chapter 6, Section 4 6.4.24
Lesson 7

Standard Form to Vertex Form—Part 1

Goal
Students will convert standard form quadratic expressions into vertex form by completing the square when the leading coefficient is 1.

Do Now (time: 5 minutes)
Using algebra tiles, students will show how to expand \((x - 3)^2\) into standard form and write it as \(x^2 - 6x + 9\). They should recognize that the rectangle they make to represent the multiplication is a square. They may not have associated the operation of squaring with the fact that it physically makes a square. This idea will be important for completing the square—you are trying to make a square.

Hook (time: 5 minutes)
With Desmos, the teacher should project a graph of \(f(x) = (x - 3)^2\) and ask students if the equation is written in vertex form. To answer this, students should notice that the vertex of the graph is at the point 3,0. This means that the value of \(H\) is 3 and the value of \(K\) is 0. On Desmos, graph \(g(x) = 1(x - 3)^2 + 0\). This now fits the form \(A(x - H)^2 + K\) more explicitly. Students should observe that the graph of \(g(x)\) is identical to the graph of \(f(x)\). Then, plot \(h(x) = x^2 - 6x + 9\) and students will see that again, the graph is identical. So we have three different ways of expressing the same function.

See: Desmos Graphing Calculator
www.desmos.com/calculator

Hook: The teacher should ask students which form they prefer—standard form or vertex form. Most students might say vertex form because it tells them so much about the graph immediately. The teacher should say:

Wouldn’t it be great if we could just change standard form to vertex form?
We can do that and we will learn how today.

Presentation (time: 20 minutes)
The teacher should use slides 1-8 of the “Algebra Tiles—Completing the Square” PowerPoint found at:

See: Algebra Tiles—Completing the Square (PowerPoint Presentation)
storeroom.norledgemaths.com/uploads/1/0/8/1/10815708/ppt_completing_the_square_1.pptx

Also at: http://bit.ly/2qp5RVb (Google Drive Math Guide resource for DYS/SEIS teachers)

Be sure to pause for sufficient time at each slide so that students can work with algebra tiles in order to address the question posed by the slide and discuss what they are doing with the teacher. The last slide gives multiple exercises which students will work on during practice and application. Do not project that slide at this time, as it gives the answers. If students need help with understanding the concepts presented, the teacher will encourage students to identify where mistakes were made in the process.

Practice and Application (time: 20 minutes)
Students will work with algebra tiles to physically make squares using the problems listed on the final slide of the “Algebra Tiles—Completing the Square” PowerPoint presentation (see above). Please
note that the slide is animated—the first click shows the problems; the second click reveals the answers. Students should rewrite each expression they work with in vertex form. They may not complete all of the exercises in the allotted time.

Review and Assessment (time: 5 minutes)
Students will respond to the following prompts through oral response or writing:

Do you think it’s true that any quadratic written in standard form can also be written in vertex form? Or does it only work for some of them? Explain your reasoning.

(Students may bring up the possibility of having an odd number as the linear coefficient in standard form or having a quadratic coefficient that is not 1.) The teacher will lead a discussion of students’ responses.

Extension
Students can work on the two challenge problems provided in the “Algebra Tiles—Completing the Square” PowerPoint presentation (see p. 6.4.25).

Lesson 8
Standard Form to Vertex Form—Part 2 (2 Days)

Note: This lesson is the second on completing the square; in this case the coefficient is not 1.

Goal
Students will convert standard form to vertex form by completing the square when the leading coefficient is not 1. They will also use vertex form to solve a profit maximization problem.

Lesson 8—DAY 1:

Do Now (time: 5 minutes)
Students will show how to convert $x^2 – 14x – 22$ into vertex form by completing the square.

Hook (time: 5 minutes)
The teacher should put standard form up on the board, with letters for coefficients so that students can refer to them by name: $ax^2 + bx + c$. Students will read what they wrote during Review and Assessment in the previous lesson and discuss it with the class. The discussion should raise the issue of a being a number other than 1 and $b$ being an odd number or a fraction. At the close of the discussion, the teacher should indicate to students that it is possible to complete the square no matter what kind of numbers they have for $a$, $b$, and $c$. Today they will learn how to do that.

Presentation (time: 20 minutes)
So far, students have completed the square when the coefficients are pretty friendly. The teacher should now ask students how they would complete the square for $x^2 – 100x + 800$. This is clearly too big to do with algebra tiles. the teacher will ask them what they would do if they had enough time and enough tiles. Students should realize that they would have to split the -100 in half, -50x for one side of the square and -50x for the other side. The teacher should review the purpose and use of algebra tiles and demonstrate to
students the idea that you always take the linear coefficient \( b \) and cut it in half, no matter what kind of number that coefficient is. There is a good physical reason for this—the square has two different regions that add up to make \( bx \). So each individual region is \( \frac{1}{2} bx \).

The teacher should then show one of the following videos on completing the square when the leading coefficient is not 1. Note: The videos use the vocabulary of “binomial” and “trinomial.”

SEE: Converting Standard to Vertex Form
www.youtube.com/watch?v=rp1iQdCCBnI (3 minutes)
Changing a Quadratic from Standard Form to Vertex Form
www.youtube.com/watch?v=dse3XdN9q2k (4 minutes)

**Practice and Application** (time: 25 minutes)
Students should convert each function from standard form to vertex form by completing the square:

\[
\begin{align*}
f(x) &= 3x^2 + 6x - 10 \\g(x) &= -2x^2 + 8x + 21 \\h(x) &= 4x^2 + 10x + 15
\end{align*}
\]

For more practice, students can use this website that allows them to check their answers:

SEE: Algebra II Exercises: Completing the Square

**Lesson 8—DAY 2:**

**Review and Assessment** (time: 5 minutes)
The teacher will first define the concept of revenue maximization: revenue is income and the task presented helps students learn how to make the income as large or as great as possible. Students will use completing the square to solve a revenue maximization problem and then they will create a maximization problem of their own. This activity should start with a short discussion about revenue (number of items sold x price of each item). The website below provides a real-world example of the use of quadratic equations in vertex form. The teacher should present the problem posed on the site and provide scaffolding as needed to help students set it up as a quadratic equation. The website offers guidance and a step-by-step process to reach a solution.

SEE: Max/Min Problem–Maximizing Revenue, Selling Calculators

To demonstrate their understanding of the problem-solving process, students will make up their own maximization problem situation. They should prepare a video or PowerPoint presentation that includes:

- A description of the situation
- An equation that models the situation, with an explanation of how they arrived at each part of that equation
- Steps for finding the maximum of the function
- A solution stated in the context of the problem

Before students begin creating their problems, the teacher should lead students in creating a simple
rubric based on these criteria. Students can then use the rubric to provide peer feedback on the problems and solutions.

**Extension**
Taking a cue from the “Min/Max Problem” website, which states, “managers at movie theaters, bus companies, airlines, video game companies, and so on use reasoning like this all the time,” students can research other real-world situations that involve finding a maximum or a minimum of some quantity using quadratic equations.

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**Lesson 9**

**The Quadratic Formula**

**Goal**
Students will observe the connection between vertex form and the quadratic formula and use the quadratic formula to solve quadratic equations.

**Do Now (time: 5 minutes)**
Students will use algebra tiles to show that \(2(x - 3)^2\) is equivalent to \(2x^2 - 12x + 18\). (Students should make two separate squares, each with a side of \((x + -3)\).)

**Hook (time: 5 minutes)**
Students will share their work from the Do Now and explain how they reached their conclusions. The teacher should explain that they will be learning a new way to solve quadratic equations: the quadratic formula.

**Presentation (time: 20 minutes)**
To show students where the quadratic formula comes from, the teacher will play the following 7-minute video. After watching the video, the teacher will ask:

- What are math terms you heard while watching this video?
- What are two things you learned from watching this video?
- What is one thing you are wonder or have a question about from watching this video?

**SEE:** Proving the Quadratic Formula
www.youtube.com/watch?v=zThTk2FbdLY

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

The teacher can also use the following website as a resource to enhance learning:

**SEE:** Cool Math: The Quadratic Formula
www.coolmath.com/algebra/09-solving-quadratics/05-solving-quadratic-equations-formula-03
Practice and Application (time: 25 minutes)
Students will practice using the quadratic formula to solve selected equations found here:

SEE: Using the Quadratic Formula

Simplifying radicals is not a high priority here. Instead, students should be able to look at a solution and estimate its size. For that purpose, the square root of 50 is actually easier to look at than 5 square roots of 2. We know that 50 is close to 49, so the square root of 50 must be a little more than 7. Students should be encouraged to estimate the value of irrational solutions and compare them against the approximate location of a root on a graph. They should use graphing software (or a graphing calculator) to help them check their answers.

Review and Assessment (time: 10 minutes)
Students will apply the quadratic formula to the high dive problem in Lesson 2, where they had originally estimated a solution by looking at the graph. They should explain in writing how they know they have a reasonable answer.

Extension
Students can look at vertex form written with literals for $A$, $H$ and $K$ and use “undoing” as a way to solve the equation. This will allow them to arrive at a quadratic formula in terms of $A$, $H$ and $K$. It’s a very simple formula!

$$H \pm \sqrt{\frac{-K}{A}}$$

Each part of the formula makes sense physically. For example, if $K$ and $A$ are opposite in sign, the graph will cross the x-axis, and real roots will exist. Also, the roots are symmetric around $H$.

CULMINATING LESSON
Includes the Performance Task, i.e., Summative Assessment—measuring the achievement of learning objectives

Lesson 10
Quadratic Functions—Matching Equations with Graphs (3 days)

Goal
After studying all aspects of quadratic functions and learning how quadratic expressions are represented in graphs, students will accurately match a set of quadratic expressions with the appropriate graphs and explain the factors that influenced their choices.

Note: This task is actually a critical thinking game, as it requires mathematical reasoning based on deep understanding of quadratic functions.
Lesson 10—DAY 1:

Do Now (time: 10 minutes)
Using words and diagrams, students will describe what general shape the graph of each of the following equations and expressions will have and explain how they know:

\[ f(x) = x^2 + 5 \quad g(x) = (x - 3)(x + 4) \quad h(x) = -1(x - 5)^2 + 2 \]

Hook (time: 15 minutes)
The teacher will project an action sequence photo like the one from the following website of a skateboarder.

SEE: Skateboarder photo

The teacher will ask students to consider whether a quadratic function is involved or not. Real-life situations are messy, and students can discuss some of the potential problems of modeling this sequence. A big problem is that the skateboarder is not a “dot,” and in fact he changes shape during the action. So if students want to measure height as a function of time, they have to decide how to measure the height. There is another big problem here in that the independent variable is time, not horizontal distance. We are relying on the assumption that the photos were snapped at equal time intervals, as in the strobe photography that students saw in Lesson 2. There also seems to be some (deliberate) distortion in the overall picture. After this discussion, students can work in pairs to make their own action sequences of any activities they like by drawing multiple still shots and then pasting them onto chart paper based on a quadratic function.

The teacher will ask students to present their finding from the Do Now, prompting them to go deeper in their explanations and underscoring what they have learned. For example, the teacher should ask students to specify, to the extent they are able, the vertex of each quadratic graph and whether the function has any zeros.

Presentation (time: 30 minutes)
The teacher should project the following list of topics from the unit and ask students something that they recall about each of them. Students will be applying their understanding of several of these topics in the summative assessment. This review may be conducted as a game, with points awarded for accuracy and clarity and opportunities for a team to capitalize by extending its opponent’s incomplete response.

- Using quadratic functions to model real-world problems
- Solving problems with graphs and with factoring
- Investigating parameters of quadratics with vertex form
- Using patterns in data tables to find the vertex and the steepness of a quadratic function
- Converting standard form to vertex form by completing the square
- Solving equations with the quadratic formula

Each of these topics has a graphing component, and the Day 2 Practice Task outlined on p. 6.4.31 will emphasize the connections between the equations and expressions and their respective graphs. If students do not readily recall key ideas from these topics or seem to be confused about several of these topics, the teacher should spend time reviewing them before moving on to preparation for the Performance Task.
Lesson 10—DAY 2:

Practice and Application (time: 55 minutes)

See the “Quadratic Functions Practice Task” Activity Sheet on pp. 6.6.11-6.6.13, which includes Graph 1.

The teacher should project Graph 1 above \((x^2 - 2x - 3)\) and ask students to state three things they know for sure about this quadratic function, based on its graph. The graph does not show its scale, but the x- and y-axes have the same scale.

Their observations could be something directly physical about the graph, such as:

- It crosses the x-axis twice.
- Or, it could be something that students conclude about the function’s equation, such as:
  - The value of the \(A\) parameter has to be positive because the U-shape is pointing upwards.

Then the teacher will show this list of equations and ask students which equation would match the graph:

\[
\begin{align*}
  f(x) &= 0.5(x - 3)^2 + 1 \\
  p(x) &= -2(x - 1)^2 - 4 \\
  g(x) &= x^2 - 2x - 3 \\
  t(x) &= 2(x - -1)^2 - 4
\end{align*}
\]

They may be able to reject \(f(x)\), \(t(x)\) and \(p(x)\) fairly quickly, since \(f(x)\) has a vertex located in Quadrant 1, \(t(x)\) has a vertex located in Quadrant 3, and \(p(x)\) has a negative value for Parameter \(A\), which makes the U-shape point downward.

But how can students know that \(g(x)\) is a good match? They have to come up with some reasons.

The teacher should tell students that this is what they will be doing in the Performance Task—matching an equation with its graph and providing evidence to support their choice.

The goal of today’s review session is to introduce students to the format of the task and to help them practice the skills they will use in order justify their matches. For each match on the Performance Task, students will fill in a chart like the one on the next page (p. 6.4.32), using the instructions that follow.
Students will use these steps to make matches on a chart like the one below:

- Fill in the left side of the chart with three facts you know about this function just by looking at the graph.
- Read the list of possible equations and pick one that matches the graph.
- Fill in the right side of the chart to show how the equation fits the things you know about the graph.

The class will fill in the chart together, using whatever ideas students come up with. The teacher will act only as recorder.

<table>
<thead>
<tr>
<th>Facts from GRAPH</th>
<th>Why the EQUATION matches</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The teacher will then hand out a copy of the completed chart from the “Quadratic Functions Practice Task” Activity Sheet located on pp. 6.6.11-6.6.13 in the Supplement. Students may be surprised to see that they can use skills like factoring and completing the square in order to justify their choice of equation to match the graph. Students should read the work and discuss it, stating what they might have done differently. The example shows one approach, but it’s certainly not the only one.

After students discuss the sample problem, the teacher will give them the next sheet with a blank chart. They should work individually to fill in the chart.

Students may use algebra tiles, graph paper, and online tools to check their work as they go through the review process:

**SEE:** Algebra Tiles (for expanding and factoring quadratic expressions)
http://illuminations.nctm.org/activity.aspx?id=3482

Desmos Graphing Calculator (for graphing equations in standard, factored, or vertex form)
www.desmos.com/calculator
LESSON PLAN

SEE: Quadratic Equations Calculator (for computing real and complex roots)
www.mathgoodies.com/calculators/quadratic_equations.html

Quadratic Formula Calculator (for graphing quadratic functions in standard form/finding real roots)
www.wolframalpha.com/widgets/view.jsp?id=f7c358d46c9ccc355492bfb66d2c59b

Lesson 10—DAY 3:

Review and Assessment (time: 55 minutes)

Students will complete the “Quadratic Functions Performance Task” Activity Sheet found on pp. 6.6.14-6.6.19 of the Supplement. An ANSWER KEY for the teacher is located on p. 6.6.20.

Each graph should be printed on its own sheet of paper, so that students can move the graphs around as they decide which equation goes with which graph (the teacher can use colored paper to help differentiate between graphs and equations). The packet should not be stapled. The equations on the first page of the Activity Sheet should be cut apart so that each equation is on a separate slip of paper, allowing students to move them around.

Note: Five graphs are provided for the Performance Task. The number of graphs used in the assessment can be adjusted at the teacher’s discretion based on students’ needs.

To help students collect their thoughts, each page asks the student to state three things they know for sure about that function. They might notice, for example, that the function has no zeros because the graph of the function never crosses the x-axis. Or, they might notice that the vertex of the graph is located somewhere in Quadrant 2. Although this is a summative assessment, it is not a conventional test, and students should continue learning in the process of working through the activities. The teacher should encourage students to use the process of inferring, stating, and explaining that they have used throughout the unit.

The criteria for evaluation of this assessment should include the following:

- Completeness and accuracy of observations listed under “Facts from the GRAPH”
- Reasoning and accuracy of explanations provided under “Why the EQUATION matches”

In addition, the following criteria from the Department of Youth Services (DYS) Future-Ready Rubric should be used to assess students’ work habits and perseverance:

- Effective Communication
- Initiative and Self-Direction
- Productivity and Accountability

Extension (if applicable)

Students who are interested in quadratic equations may wish to read about the roles they have played in human history from the ancient Babylonians to the present day: 101 Uses of a Quadratic Equation, Part I and 101 Uses of a Quadratic Equation, Part II.

SEE: 101 Uses of a Quadratic Equation, Part I
https://plus.maths.org/content/101-uses-quadratic-equation

101 Uses of a Quadratic Equation, Part II
https://plus.maths.org/content/101-uses-quadratic-equation-part-ii?src=aop
POST-UNIT REFLECTION

On meeting the Learning and Language objectives
Connections to Empower Your Future
UNIT: Quadratic Functions

Future Ready Connections

This unit encourages students to activate their Future Ready skills, think creatively and critically, and to reflect on their hypotheses and results. Throughout the unit, students are encouraged to experiment, reflect, and try new approaches to solving problems and using equations and tools. 

This freedom to experiment will give teachers the opportunity to evaluate students on their initiative, self-direction, and accountability for completing tasks, actively engaging in critical thinking, and taking responsibility for their own discoveries. Teachers should reflect on whether or not youth stay on task without prompting, and if they push themselves to thoroughly complete each activity, answer their own questions, and create a detailed final product instead of only addressing the minimum required information or waiting for explanations from the teacher. Youth have many opportunities to strengthen their communication and listening skills through group discussions and the presentation of their projects in Lesson 3 and Lesson 8. Students will also communicate through writing in Lesson 7 which can be evaluated for clarity and effectiveness.

Teachers are encouraged to use the Future Ready Rubric to evaluate students’ growth and are encouraged to have students self-evaluate their progress using the Future Ready Rubric. Students should reflect on how they demonstrated growth, what they could do to further improve their skills, and how they are transferable to other situations and experiences.

Transfer Goal and Essential Question Connections

One Transfer Goal for this unit states that students will use quadratic equations to solve real-world problems. This unit strongly emphasizes that students can and will see quadratic functions in the real world, which should encourage youth to see the relevance of the topic and the importance of mastering the skill of graphing and experimenting with equations and outcomes. Lesson 2 asks students to analyze the diving function and corresponding graphs to analyze a diver’s arc through the air when diving into water. Lesson 8 has students use the real-world example of revenue for a company selling calculators to analyze the use of quadratic equations in vertex form. Teachers should consider expanding on this Transfer Goal by having youth analyze or create additional situations that can be measured with quadratic equations and graphs. Some ideas include: planning a position and timing for fireworks display, throwing a ball, creating and selling something for a profit (manufacturing costs, demand, and profit).

Career Exploration Connections

The extension activities in Lesson 8 encourage students to research other real-world situations that involve finding a maximum or a minimum of some quantity using quadratic equations. Suggestions include considering how managers at movie theatres, bus companies, airlines, or video game companies use quadratic equations. Teachers should consider expanding on this idea by having youth research careers and industries that use algebra and quadratic equations. Students can brainstorm additional careers such as registered nurses, economists, medical researchers, engineers, farmers, environmental conservationists, factory workers, information and record clerks, real estate agents, insurance agents, and military personnel. Students can also identify specific tasks within a job that require the use of quadratic equations and
Teachers should encourage students to see that knowledge of algebra and specifically quadratic functions does play a role outside of the classroom and in many career fields."

an understanding of algebra. Some examples include: automotive engineers use them to design brake systems, audio engineers use equations to design sound systems that have the best sound quality possible, astronomers use quadratic equations to describe the orbits of planets, and human resources workers use equations to figure out how to design and pay for pension plans. Teachers should encourage students to see that knowledge of algebra and specifically quadratic functions does play a role outside of the classroom and in many career fields.

**PYD/CRP Connections**

This unit reflects many aspects of Culturally Responsive Practice and Positive Youth Development by utilizing familiar and interesting examples, and by allowing youth to have a voice as they experiment and grow in many areas. The use of OK Go’s video “Upside Down and Inside Out” is a powerful hook to engage students and demonstrate that learning about math and using quadratic functions can be entertaining and useful. Other real-world examples throughout the unit emphasize how students are impacted by and see quadratic equations in their daily lives. This helps youth create meaningful connections to the topic and deepen their understanding of how this knowledge appears in the world.

*The unit also allows for exploration, experimentation, reflection, and scaffolding so that students are active participants in their learning and are a part of the discovery process.*

This is especially evident in Lesson 2 when students must choose a scale to use in their graph and will need to experiment and try many scales to best represent the data. By encouraging youth to take risks and to be part of the discovery process, teachers are grounding the tasks in the students’ strengths and allowing students to have a voice in the classroom and in their own learning. The lessons allow for differentiation (specifically noted in Lesson 4’s Review and Assessment) so that all youth can rely on their strengths and take on appropriate challenges that nurture their development. Teachers are encouraged to differentiate lessons and engage youth as resources in the classroom.

For Technical Assistance with Empower Your Future connections and lessons, please request support by submitting a Coaching Request ticket using the Coaching Feature on TeachPoint.
Investigating Quadratics in Vertex Form
Lesson 4

DIRECTIONS: For this activity, you will use the Desmos online graphing calculator.
SEE: Desmos Graphing Calculator | www.desmos.com/calculator

1. At the left of the Desmos screen, you should enter the equation for a quadratic function.

Try this equation first:

\[ f(x) = 0.4(x - 3)^2 + 1 \]

To access the squaring feature, use the \( a^2 \) button on the Desmos keypad.

Draw a sketch of \( f(x) \) on the graph at right.

Keep \( f(x) \) on the screen so you can compare it to this one:

\[ g(x) = -0.4(x - 3)^2 + 1 \]

Draw a sketch of \( g(x) \) on the graph at right.

a. What is the only difference in the two equations?

b. What difference does that make in how the curves look?
2. Here is a graph that looks like a smile. This graph is considered to be “right side up.”

The lowest point on a curve is called a “minimum.” What are the coordinates of the minimum in the graph?

(____ , ____)

The next graph looks like a frown. It is considered to be “upside down”.

The highest point on a curve is called a “maximum.” What are the coordinates of the maximum in the graph?

(____ , ____)

3. All of your equations will be written in this form:

\[ f(x) = 1.6 \ (x - 5)^2 + 3 \]

You will always use the parentheses and the squaring. Choose numbers to put in the blanks.

In this equation, for example, 1.6 is used for the first blank, 5 for the second blank and 3 for the third blank.

The equation would be entered into Desmos as: \( f(x) = 1.6(x - 5)^2 + 3 \)

Try that and see what the curve looks like. You can use the “X” button on the right side of an equation to clear away any other curves you have drawn previously.

4. The blanks in the equation have labels. The first blank is called Parameter \( A \), the second one is Parameter \( H \) and the last one is Parameter \( K \).

Keep \( f(x) = 1.6(x - 5)^2 + 1 \) on the screen. Then, make a new curve for \( g(x) \).

Use 1.6 for Parameter \( A \) and 3 for Parameter \( K \), but change the value of Parameter \( H \) to a negative number.

\[
\begin{align*}
\frac{f(x)}{A} &= \frac{(x - H)^2 + K}{K} 
\end{align*}
\]

Compare \( g(x) \) to \( f(x) \).
What happened to the graph when you changed the \( H \) value to a negative number?

5. You will always compare the width of curves by looking at the vertex. The vertex is the point where the graph changes direction, so it is either at the minimum or at the maximum of the graph.

Here are two graphs that are in different positions. Which graph is wider at its vertex?
6. Clear the screen. Enter this equation into Desmos Graphing Calculator and look at the graph.

\[ t(x) = 1(x - 5)^2 + 2 \]

This curve is considered to be “normal” in width. Keep it on the screen.

Next, put in \( q(x) = 0.4(x - 5)^2 + 2 \). How does the width of \( t(x) \) compare to \( q(x) \) when you look at the vertex?

Delete \( q(x) \) from the screen. Put in \( c(x) = 3(x - 5)^2 + 2 \). How does the width of \( c(x) \) compare to \( t(x) \) when you look at the vertex?

7. Clear the screen. Enter \( f(x) = 1(x - 2)^2 + 1 \) into Desmos Graphing Calculator. Now make a new curve \( g(x) \) which is thinner than \( f(x) \) and has its minimum at (-3, -4). Write the equation for \( g(x) \) here:

8. Clear \( g(x) \) from the screen. Then make a new curve \( r(x) \) which is wider than \( f(x) \) and has its maximum point located directly on the x-axis. Write its equation here:

9. What do you have to do to the equation in order to make the curve go upside down?
10. At right, you see a picture of the graph of \( g(x) \). Your job is to use Desmos to find an equation that will match the graph. Write the equation here:

\[
g(x) = \]

11. You can see on the picture that the graph passes through the point (-5, 0). If your equation is correct, then it should be true that \( g(-5) = 0 \). Demonstrate here that your equation works to make \( g(-5) = 0 \):

12. How does it change the graph if you use a positive number for parameter \( K \) versus a negative number for \( K \)?

13. Summarize how you think changing \( A \), \( H \), and \( K \) affect the way the graph looks. What is the job of each parameter?

\[ A: \]

\[ H: \]

\[ K: \]
14. Here is a graph that hits the point $(2, 5)$. It is on the “side” of the graph, not at the vertex.

Design a function $f(x)$ that goes through the point $(6, 1)$ somewhere along the side of the curve (but not at the vertex). You may have to zoom in several times to make sure that you are really hitting the point. Write its equation here:

$$f(x) =$$

15. Now use your equation to show that $f(6)$ definitely equals 1.

16. Clear the screen. Enter this equation:

$$m(x) = 0.1(x - 3.6)^2 - 2$$

Now make another curve $r(x)$ which has its minimum at the same location, but make it wider and more open at the vertex. Write its equation here:
17. Clear the screen and enter \( f(x) = -2(x - 3)^2 + 8 \). This function has two zeros—where are they? Use the equation to show that \( f(x) = 0 \) for each of those \( x \) values.

18. Next change the equation to \( f(x) = -2(x - 3)^2 + 6 \). This function also has two zeros but they are not integers. Estimate them from the graph.

19. We can find the values of those zeros exactly by using inverse operations. Call your teacher over to discuss how to solve this equation. The first step is provided for you.

\[
-2(x - 3)^2 + 6 = 0
\]

\[
-2(x - 3)^2 = -6
\]

When you get your solutions, check to see if those numbers are close to the estimate you made in #18.

20. In a previous math class, you may have learned about reflection symmetry. A shape has reflection symmetry if you can find a way to fold it in half so that all of the parts match up exactly. The place where you fold it is called the line of symmetry.

Clear the screen and enter this equation:

\[
r(x) = 0.9(x - 3)^2 + 4
\]

This curve has its line of symmetry at the point where \( x = 3 \). Enter “\( x = 3 \)” into Desmos and you will see that line.

Keep \( r(x) \) on the screen. Now make another curve \( b(x) \) which also has its line of symmetry at \( x = 3 \), but the vertex is lower down on the screen. Write its equation here:
21. Now design a curve $d(x)$ which has its line of symmetry directly along the $y$-axis. Write its equation here:

22. Design a curve $t(x)$ which will never go into Quadrant 1, no matter how much you zoom out. This is a tricky problem. You might want to start by making a sketch on paper before you use Desmos. When you get it to work, write the equation here:

23. Explain how you know for sure that $t(x)$ will never go into Quadrant 1.

24. Clear the screen and enter these equations:

   $f(x) = .8(x - 3.2)^2 - 5.4$

   $g(x) = .9(x - 3.2)^2 - 5.4$

Keep both curves on the screen.

Next, design a curve that’s in between the other two—steeper than $f(x)$ and flatter than $g(x)$. Write its equation here:
Investigating Quadratics in Vertex Form
Lesson 4

25. Clear the screen and enter this equation:

\[ p(x) = 1.8(x - 0.5)^2 + 2 \]

It appears that the curve for \( p(x) \) goes through the point (2, 6). But we don’t know for sure if it goes through that point EXACTLY until we use the equation to see if \( p(2) = 6 \). Get a calculator. What value do you get for \( p(2) \)?

26. Change only one parameter in the equation for \( p(x) \) so that the graph really goes through (2, 6). Demonstrate here that your equation definitely works.

27. Clear the screen. Make a curve \( c(x) \) which goes through the point (3, 1) and also goes through the point (1, 3). This is tough. Just keep adjusting your parameters until you get it right. Use a recording sheet to keep track of what you try. Write your best equation here:

28. Does \( c(x) \) go through (1, 3) and (3, 1) exactly, or is it just close? Say how you know.

29. Clear the screen. Now make a curve which goes through the points (-5, -1) and (5, -1). Use a recording sheet to keep track of what you try. This time you have to find an equation that hits both points exactly.
CHALLENGE PROBLEMS:

30. Look for other solutions to problem #28. There are many possibilities, all of which work exactly. List your equations here:

31. Do you think it’s true that for any two points your teacher assigns you, it is possible to put a curve through them exactly? Any two at all? Do several experiments to investigate this. Explain your thinking here:

32. Now consider the same question for any group of three points that your teacher might pick. No matter what points are picked, can you put a curve through them? Explain your thinking here:
DIRECTIONS: This activity is a review in preparation for the Performance Task. After going over the information on this page, you will complete the problem on page 3 of this Activity Sheet. An example of a completed problem is shown on page 2.

Possible equations for Graphs 1 and 2 are shown below. Your task is to match each graph with the correct equation.

\[ f(x) = \frac{1}{2}(x - 3)^2 + 1 \]
\[ p(x) = -2(x - 1)^2 - 4 \]
\[ g(x) = x^2 - 2x - 3 \]
\[ t(x) = 2(x - 1)^2 - 4 \]

Follow these steps to make your match and complete the problem on page 3 of this Activity Sheet:

- Fill in the left side of the chart at the bottom with three facts you know about this function just by looking at the graph.
- Read the list of possible equations and pick one that matches the graph.
- Fill in the right side of the chart to show how the equation fits the things you know about the graph.

The chart for Graph 1 has been filled in for you as an example. After you read the example and discuss it, you will fill in the chart for Graph 2, on the last page of this Activity Sheet. This will help you get ready for the Performance Task.

Note: There are no tick marks on the x or y axis, so you don’t know what the scale is here. However, all of the graphs use the same scale. This lets you compare them to each other.
### Quadratic Functions Practice Task
Lesson 10

**GRAPH 1**

**SAMPLE SOLUTION:**
This example has been filled in. Now, it’s your turn to fill in the worksheet Graph 2, on the next page.

**Matching Equation:**
\[ g(x) = x^2 - 2x - 3 \]

<table>
<thead>
<tr>
<th>Facts from the GRAPH</th>
<th>Why the EQUATION matches</th>
</tr>
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</table>
| The y-intercept is negative. | \[ g(x) = x^2 - 2x - 3, \]  
so if you make \( x = 0 \), you get this:  
\[ g(0) = 0^2 - 2 \cdot 0 - 3 = -3 \] |
| It has two real roots—one positive and one negative. The negative root is closer to zero than the positive root is. | To find the roots, I could use the quadratic formula, but then I noticed that this one can be written in factored form:  
\[ (x + 1)(x - 3) \text{ is equivalent to } x^2 - 2x - 3 \]  
So \( g(x) \) will = 0 when \( x = -1 \) and also when \( x = 3 \). That goes with what I said about the roots. |
| The vertex is in quadrant 4. | I will complete the square to get this into vertex form:  
\[ g(x) = x^2 - 2x - 3 \]  
\[ g(x) = x^2 - 2x + 1 - 4 \]  
\[ g(x) = (x - 1)^2 - 4 \]  
See the vertex is at the point (1,-4) and that’s in quadrant 4! |

Graphs generated by Desmos. Permission for use granted by Desmos.
### Quadratic Functions Practice Task

**Lesson 10**

**Matching Equation:**

**GRAPH 2**

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<th>Facts from the GRAPH</th>
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Graphs generated by Desmos. Permission for use granted by Desmos.
### Quadratic Functions Performance Task

**Lesson 10**

**DIRECTIONS:** Cut out the equations below (or fold and tear the sheet) so you can match them with the graphs they represent that follow in this Activity Sheet. Note: Not all of these equations will match a graph. There are some extras here that don’t go with anything.

<table>
<thead>
<tr>
<th>Equation</th>
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<tbody>
<tr>
<td>$b(x) = (x + 3)(x + 5)$</td>
<td>$p(x) = 1.5(x - 4)^2 + 2$</td>
</tr>
<tr>
<td>$f(x) = -0.5(x - 4)^2 + 2$</td>
<td>$q(x) = (x - 3)(x - 5)$</td>
</tr>
<tr>
<td>$g(x) = 0.2(x - 3)^2 + 5$</td>
<td>$r(x) = (x - 2)^2$</td>
</tr>
<tr>
<td>$j(x) = x^2 + 4$</td>
<td>$t(x) = -3(x - 4)^2 + 2$</td>
</tr>
<tr>
<td>$m(x) = x^2 + 5x - 6$</td>
<td>$z(x) = -x^2$</td>
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</table>
DIRECTIONS:
Follow steps 1-3 below to make your matches for each graph.

Note:
There are no tick marks on the x or y axis in the graphs, so you don’t know what the scale is here. However, all of the graphs use the same scale, which lets you compare them to each other.

1. Fill in the left side of the chart with three facts you know about this function just by looking at the graph.

2. Look at all of the possible equations on Page 1 of the Activity Sheet and pick one that matches the graph.

3. Fill in the right side of the chart to demonstrate how the equation fits the things you know about the graph.

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<thead>
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Quadratic Functions Performance Task
Lesson 10

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Quadratic Functions Performance Task
Lesson 10

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Quadratic Functions Performance Task
Lesson 10

**GRAPH 4**

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# Quadratic Functions Performance Task

**Lesson 10**

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Graphs generated by Desmos. Permission for use granted by Desmos.
ANSWER KEY
Lesson 10—Performance Task

GRAPH 1: \( g(x) = 0.2(x - 3)^2 - 5 \)

GRAPH 2: \( p(x) = 1.5(x - 4)^2 + 2 \)

GRAPH 3: \( b(x) = (x + 3)(x + 5) \)

GRAPH 4: \( r(x) = (x - 2)^2 \)

GRAPH 5: \( f(x) = -0.5(x - 4)^2 + 2 \)
Introduction

When students are first introduced to the idea of “imaginary” numbers, they wonder what practical purpose can come from studying something that isn’t “real.” They may meet teachers with resistance since they do not think something “imaginary” is worth studying. However, after studying this unit, we hope that students will begin looking at mathematics not as a set of fixed rules to be memorized, but instead as a human endeavor that is always evolving.

The “Complex Numbers” unit is designed to come first in the Winter Season of Advanced Functions and will take two weeks to complete. Teachers in long-term programs have the option of extending the unit into a three-week unit by using the extension activities that are provided after most lessons. The Performance Task can also be extended from writing a book proposal to writing a book called Complex Numbers for Beginners. The extended Performance Task will require students to write more extensive explanations and draw more detailed illustrations; both will deepen their understanding of the concepts discovered in the unit.

To truly deepen students’ understanding of concepts, the “Complex Numbers” unit employs an inquiry model that draws on students’ natural curiosity and inferencing skills to develop and test hypotheses. Students will be given many opportunities to make their thinking visible and will be given a variety of customizable tools to help them understand important concepts. Teachers will focus discussions around the Essential Questions in order to encourage students to think more deeply about the topics being studied.

Why do mathematicians make up new kinds of numbers?
Why are complex numbers useful?

While there are many big ideas addressed in this unit, the “Complex Numbers” unit specifically addresses these three standards.

N-CN.1: Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with $a$ and $b$ real.

N-CN.2: Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
**N-CN.7**: Solve quadratic equations with real coefficients that have complex solutions.

The first of these standards, **N-CN.1**, requires that students accept that there can be a square root of a negative number.

Once students are able to grasp this concept, the remaining two standards are easier for students to understand since they require students to apply operations they already know to this new category of numbers.

It is understandable that students might have difficulty grasping the concept of this new category of an “imaginary” number, but it is important for students to realize that “imaginary” numbers are just as real as “real” numbers. To help students understand this concept, it is useful for teachers to encourage students to view complex numbers as a location on a number plane rather than a number line. Furthermore, a solid foundation from previous geometry and algebra classes will help students be successful in this unit. Having prerequisite knowledge such as being able to locate points on a coordinate plane and being able to use the Pythagorean theorem are essential to understanding the concepts encountered here. Students should also be familiar with the quadratic formula and be able to solve higher-degree polynomial equations. If students do not have adequate skills, the teacher will need to provide time for students to master these prerequisites.

For adaptation ideas for this unit, see p. 6.7.3 on the right.
# Advanced Functions: Complex Numbers

**Adapting This Short-Term Unit for Long-Term Programs**

## Plan 1 (Short)

### WINTER SEASON—Complex Numbers: Short-Term Programs

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<tr>
<th>MONDAY</th>
<th>TUESDAY</th>
<th>WEDNESDAY</th>
<th>THURSDAY</th>
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<tr>
<td>Week 1</td>
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<tr>
<td><strong>Lesson 1:</strong> Introduction to Complex Numbers</td>
<td><strong>Lesson 2:</strong> Adding, Subtracting, and Multiplying Complex Numbers</td>
<td><strong>Lesson 3:</strong> Powers of Complex Numbers</td>
<td><strong>Lesson 4:</strong> Julia Sets</td>
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<tr>
<td>Week 2</td>
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<td><strong>Lesson 5:</strong> Using the Quadratic Formula to Solve Equations with Complex Roots</td>
<td><strong>Lesson 6:</strong> Finding Roots in the Complex Plane—Part 1</td>
<td><strong>Lesson 7:</strong> Finding Roots in the Complex Plane—Part 2</td>
<td><strong>Lesson 8:</strong> Complex Numbers for Beginners Performance Task (minimized version)</td>
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## Plan 2 (Long)

### WINTER SEASON—Complex Numbers: Long-Term Programs

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<tr>
<th>MONDAY</th>
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<tr>
<td>Week 1</td>
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<tr>
<td><strong>Lesson 1:</strong> Introduction to Complex Numbers</td>
<td><strong>Lesson 2:</strong> Adding, Subtracting, and Multiplying Complex Numbers (with additional time for Extension)</td>
<td><strong>Lesson 3:</strong> Powers of Complex Numbers</td>
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<td>Week 2</td>
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<tr>
<td><strong>Lesson 4:</strong> Julia Sets</td>
<td><strong>Lesson 5:</strong> Using the Quadratic Formula to Solve Equations with Complex Roots</td>
<td><strong>Lesson 6:</strong> Finding Roots in the Complex Plane—Part 1</td>
<td><strong>Lesson 7:</strong> Finding Roots in the Complex Plane—Part 2</td>
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<td>Week 3</td>
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<tr>
<td><strong>Lesson 8:</strong> Complex Numbers for Beginners Performance Task (extended version)</td>
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This unit should be suitable for most short-term programs, but it would be very helpful to students entering mid-unit to keep artifacts from each lesson posted in the room as anchor charts. If the unit must be shortened, the two-day lesson on Julia Sets (Lesson 4) may be omitted, but this lesson is highly motivating for students with an artistic bent and should be retained if possible. Lessons 6 and 7, focusing on finding complex roots in higher-degree polynomials, go beyond the standards for the unit, but since this unit follows polynomial functions in the scope and sequence, they connect to previous learning.

For use in long-term programs, the unit may be extended by pursuing the suggestions provided in the extensions according to students’ interests and strengths. The teacher may also wish to extend the Performance Task by asking students to create the “Complex Numbers for Beginners” book rather than just a book proposal. To do so would require students to write more extensive explanations and provide more detailed illustrations, but the process would certainly deepen their understanding. Since this unit draws extensively on prior algebra and geometry knowledge and skills, the teacher may need to extend it to review concepts and operations that students have forgotten or never fully understood such as simplifying radicals, the Pythagorean theorem or the quadratic formula.
Emphasized Standards *(High School Level)*

**NUMBER AND QUANTITY**

**N-CN.1:** Know there is a complex number such \( i \) that \( i^2 = -1 \), and every complex number has the form \( a + bi \) with \( a \) and \( b \) real.

**N-CN.2:** Use the relation \( i^2 = -1 \) and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

**N-CN.7:** Solve quadratic equations with real coefficients that have complex solutions.

**Essential Questions** *(Open-ended questions that lead to deeper thinking and understanding)*

Why do mathematicians make up new kinds of numbers?

Why are complex numbers useful?

**Transfer Goals** *(How will students apply their learning to other content and contexts?)*

Students will articulate that mathematics is a human creation that evolves as needed to address inconsistencies.

Students will demonstrate how complex numbers can be used to plot directions and create art.

Students will explain and illustrate abstract concepts and multi-step operations to non-expert audiences.
### Complex Numbers

Complex numbers are composed of two parts, real and imaginary. Complex numbers were invented to address a closure issue with the operation of square-rooting on the set of real numbers. Multiplying complex numbers has two effects: scaling and rotating.

#### Magnitude

The magnitude of a complex number measures its distance from the origin on the complex plane.

#### Julia Set

Repeated operations on complex numbers can be used to create abstract art.

#### Audience (for writing and presentation)

The organization, content, and style of a piece or writing or presentation must be adapted to the audience and purpose.

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<table>
<thead>
<tr>
<th>Students should know...</th>
<th>understand...</th>
<th>and be able to...</th>
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<tbody>
<tr>
<td>Complex number</td>
<td>Complex numbers are composed of two parts, real and imaginary. Complex numbers were invented to address a closure issue with the operation of square-rooting on the set of real numbers. Multiplying complex numbers has two effects: scaling and rotating.</td>
<td>Locate complex numbers on the complex plane. Add, subtract and multiply complex numbers using the commutative, associative, and distributive properties. Use complex numbers to solve quadratic and higher degree polynomial equations.</td>
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<tr>
<td>Complex plane</td>
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<td>Find the magnitude of a complex number. Identify how various operations affect the magnitude of a complex number.</td>
</tr>
<tr>
<td>Magnitude</td>
<td>The magnitude of a complex number measures its distance from the origin on the complex plane.</td>
<td>Create Julia Set images using an online generator and explain how they are formed.</td>
</tr>
<tr>
<td>Julia Set</td>
<td>Repeated operations on complex numbers can be used to create abstract art.</td>
<td></td>
</tr>
<tr>
<td>Audience (for writing and presentation)</td>
<td>The organization, content, and style of a piece or writing or presentation must be adapted to the audience and purpose.</td>
<td>Present explanations and illustrations (including graphs) of complex number operations comprehensible to non-experts.</td>
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</tbody>
</table>
Performance Task and Summative Assessment (see pp. 6.8.27-6.8.30)

Aligning with Massachusetts standards

Using the GRASPS format, the students will create a proposal for a “Complex Numbers for Beginners” book that explains the concept of complex numbers, why people need to understand them, how to perform operations with them, and to represent them graphically (Lesson 8).

Students will present using a variety of formats including PowerPoint, Prezi, posterboard, and an oral presentation. Criteria will include, but is not limited to:

- Why a “Complex Numbers for” book is needed
- An overview of contents in the book including illustrations
- A definition of complex numbers, an illustration of how complex numbers are graphed on a complex plane, and how the Pythagorean theorem can be used to calculate their magnitude
- A demonstration of how complex numbers are added, subtracted, multiplied, and divided and how these operations are reflected in movement on a complex plane
- An explanation and demonstration of how to solve quadratic equations with real coefficients that have complex solutions

Pre-Assessment (see pp. 6.8.9-6.8.11)

Discovering student prior knowledge and experience

Lesson 1: Students will state, orally or in writing, what it means to evaluate the expression $(3 – 4)$ and then brainstorm how they might explain its meaning to a small child.

Formative Assessments (see pp. 6.8.9-6.8.27)

Monitoring student progress through the unit

Lesson 1: Students will write in response to this prompt:

If you were trying to add the complex number $(2, 7)$ or $2 + 7i$ with the complex number $(3, -5)$ or $3 – 5i$, what answer do you think you should get? Explain your reasoning.

Lesson 2: Students will conduct mathematical operations with complex numbers that have both real and complex results, and they will design problems for which the results are real and complex.
Lesson 3: Students will raise a complex number by powers of 2, 3, 4, etc., showing the multiplication they use and plotting each result on the complex plane. Students will observe how the multiplication affects both magnitude and angle, and use that pattern to make predictions.

Lesson 4: Students will create Julia set images using an online generator and explain orally how the formula $f(z) = z^2 + c$ is used to create the images.

Lesson 5: Students will apply the quadratic formula to solve equations with complex roots and accurately explain the steps in the process.

Lesson 6: Students will use 1-inch chart paper to make a poster to show a solution for the equation $x^3 = -8$. Using a protractor to measure a 60 degree angle and a ruler to measure a distance of 2 inches, students should mark the location of the root in Quadrant 1. They should also show where $x^2$ would be and finally where $x^3$ would be. The scaled diagram should demonstrate that students understand the rotation and stretching that occur as $x$ is raised to a power.

Lesson 7: Students will make up three different polynomial equations: a cubic equation that has three real roots, a cubic equation that has one real root and two complex roots, and a quartic equation that has two real roots and two complex roots. For each equation, students will explain orally how they designed it and how they know it works.
Multiple Means of Engagement

This is the why of learning, what makes students engage or disengage. Throughout the unit plan, the student will be provided with as many choices in the level of challenge and complexity as possible in order to recruit and sustain engagement. For example, the teacher will encourage and support students in setting their own personal, academic, and behavioral goals. The teacher will use many strategies to guide students, including reminders, guides, rubrics, checklists, and prompts among other things that focus students on self-regulatory goals. Student tasks will be varied, allowing for active participation, exploration, and experimentation. The teacher will provide differentiated models, scaffolds, and feedback, as well as content material that is culturally relevant and responsive to student’s backgrounds. Most important is that teachers design assignments and tasks with authentic outcomes, and that are purposeful and convey meaning to real audiences.

While complex numbers might seem too abstract to engage student interest, the unit includes real-world applications of these concepts, ranging from changing the course of a boat (Lesson 2) to creating abstract art with Julia Sets (Lesson 4). More important, the unit employs an inquiry model that draws on students’ natural curiosity and powers of inference to develop and test hypotheses. Lessons include the use of interactive online tools that streamline repetitive calculations.

Multiple Means of Representation

This is the what of learning. There are many pathways to conveying information to students. Throughout the unit, the teacher will provide information and materials in several modalities such as diagrams, vocabulary cards, and word walls, posters, and charts with formulas for calculations; and models, videos, and audio for text. The teacher will also demonstrate concepts through hands-on activities.

This unit places particular emphasis on the relationships between mathematical equations and expressions and the graphs that are used to model them, including graphs on the complex plane, which make otherwise abstract concepts more concrete. Avoiding textbooks entirely, all lessons in this unit relies on a variety of “texts” (including websites, videos, and graphing/calculating tools) to represent key concepts, many of them generated by students themselves on paper and online.

Multiple Means of Action and Expression

This is the how of learning. In learning activities students will be provided options for demonstrating what they know and can do. Students will have access to assistive technology and use multiple media.

For Empower Your Future Connections, see p. 6.9.1
For example, students will have access to word processors with grammar checks, word prediction, and spell checkers. Students could complete projects by making PowerPoint presentations, rapping through music videos, or drawing illustrations. In addition, students will have access to calculators. The teacher will scaffold writing or composing activities using tools such as concept maps, outlining tools, or graphic organizers. Students may need sentence starters and story webs to complete writing or composing tasks. The teacher will also break down long-term goals into short-term reachable goals.

Students will have a variety of opportunities to express what they have learned and are learning in this unit, including informal writing and graphing (on paper and online); arithmetic, geometric, and algebraic computations; and especially talk: conversations among peers, Socratic interactions with the teacher, and a final formal presentation accompanied by PowerPoint slides or poster board.

**Literacy and Numeracy**

**Across Content Areas**

**Reading**

Students will have to read internet articles, take notes, and discuss concepts with peers. They will also “read” and interpret algebraic expressions and equations and their corresponding graphs, including graphs on the complex plane.

**Writing**

Students will write guided notes, Exit Tickets, and other writing-to-learn activities explaining their understandings of key concepts and speculating about problems not discussed in class through inference and analysis.

**Speaking and Listening**

Students will contribute to class discussions of key concepts by making inferences from class presentations, stating their hypotheses, and explaining their reasoning. They will also make formal presentations of their work to peers.

**Language**

Students will learn and use appropriately vocabulary related to complex numbers.

**Numeracy**

Students will employ basic algebraic operations using the commutative, associative, and distributive properties as well as the Pythagorean theorem when working with complex numbers. They will apply concepts and operations from previous Algebra 2 units, including quadratic and polynomial functions. Students will also use inference and interpretation when creating and interpreting graphs.
Resources (in order of appearance by type)

Websites

Lesson 1
betterexplained.com/articles/a-visual-intuitive-guide-to-imaginary-numbers/

www.sjsu.edu/faculty/watkins/complex.htm

www.youtube.com/watch?v=EQviquyrDxA

Lesson 2
betterexplained.com/articles/a-visual-intuitive-guide-to-imaginary-numbers/

Lesson 3
www.youtube.com/watch?v=86FmwaaDH60

Lesson 4
www.zazzle.co.uk/julia+set+posters

www.youtube.com/watch?v=gruJ0S3TTtI

www.youtube.com/watch?v=w-nlYBJeErk

www.youtube.com/watch?v=2AZYZ-L8m9Q

www.mathisfun.com/numbers/complex-number-calculator.html

www.wolframalpha.com/input/?i=complex+numbers

www.easyfractalgenerator.com/julia-set-generator.aspx

Lesson 5
www.desmos.com/calculator

Lesson 6
www.youtube.com/watch?v=86FmwaaDH60

www.wolframalpha.com

Lesson 7
www.wolframalpha.com
Lesson 8
www.dummies.com/how-to/content/how-to-perform-operations-with-complex-numbers.html

www.mathsisfun.com/numbers/complex-number-calculator.html

www.wolframalpha.com/input/?i=complex+numbers

www.youtube.com/watch?v=id5JfrSD1i0  (clip from the movie, Dimensions)

Materials

1-inch graph chart paper

Lesson 3: Complex Number Designs Activity Sheet p. 6.10.1
PREREQUISITES: Math skills needed for this unit

*Complex Numbers* is the first unit in the Winter Season of Algebra 2. The prerequisite math skills summarized below are taught in units that precede this unit (see Scope and Sequence chart). The following skills will be needed for students to successfully complete the this unit.

Students should know:

- The commutative, associative, and distributive properties
- How to use the quadratic formula to solve quadratic equations
- How to recognize the general shape of graphs for higher-degree polynomials
- How to connect the degree of a polynomial to the number of zeros
- How to divide polynomials (using a rectangular multiplication diagram)
- How to use the rational root theorem to identify potential roots of a polynomial equation
- How to solve polynomial equations (real roots only)
- How to connect the zeros of a function to roots of the equation where \( f(x) = 0 \)
- Key vocabulary: *term*, *coefficient*, *exponent*, *degree*, *zero*, *root*, *cubic*, *quartic*

Outline of Lessons

Introductory, Instructional, and Culminating tasks and activities to support achievement of learning objectives

**INTRODUCTORY LESSON**

*Stimulate interest, assess prior knowledge, connect to new information*

**Lesson 1**

Introduction to Complex Numbers

**Goal**

Students will articulate that mathematical abstractions such as negative numbers and imaginary numbers are human creations that are created as needed to solve problems. They will also see complex numbers as two-dimensional numbers that can be plotted on a number plane, with a measurable distance from the origin.
Do Now (time: 5 minutes)
Students will state, orally or in writing, what it means to evaluate the expression \((3 - 4)\) and then brainstorm how they might explain its meaning to a small child.

Hook (time: 10 minutes)
The teacher and students will read together “Really Understanding Negative Numbers” sections of the Better Explained website. (scroll down on website) The teacher wants students to understand that negative numbers were once considered absurd, but now are commonplace in mathematics. This realization will lead into the presentation on imaginary numbers, a more recent mathematical invention. The teacher could have students jigsaw the article and then facilitate a class discussion using the following guiding questions.

1. What do you think an imaginary number is?
2. What is the connection between negative numbers and imaginary number?

See: A Visual, Intuitive Guide to Imaginary Numbers
betterexplained.com/articles/a-visual-intuitive-guide-to-imaginary-numbers

Presentation (time: 20 minutes)
To introduce imaginary numbers, the teacher should play the first five minutes of the video “Understanding Imaginary Numbers” on the Better Explained website above, stopping when the speaker reaches “Finding Patterns.” Then the teacher should review the “Enter Imaginary Numbers” and “Visual Understanding of Negative and Complex Numbers” sections of the website, modeling on the board notes and diagrams that students will write and draw at their desks to demonstrate their understanding of the concept that multiplying by \(i\) is represented by a 90 degree rotation on the complex plane through note taking and/or illustration. Students will turn and talk to discuss connections between their notes and/or illustrations.

Once students are comfortable with these ideas, the teacher will introduce complex numbers, which are presented in a later section of the website, “Understanding Complex Numbers.” The emphasis for this lesson should be on the idea of two-dimensional numbers. Up to now, students have worked with one-dimensional numbers plotted on a number line. Now they are expanding the universe to include two-dimensional numbers plotted on a number plane. It allows them to pack two pieces of information into one number. Typically students have a very hard time swallowing the idea of “imaginary” numbers.

Note: “Imaginary” is actually a very unfortunate term. Both parts of a complex number are equally real. As the author of the website argues, students will accept complex numbers much more easily if they see the geometric interpretation of them first. The notation of “\(i\)” can be attached to that idea after the physical understanding has been established. Complex numbers can be written as ordered pairs.

See: The Nature of Complex Numbers
www.sjsu.edu/faculty/watkins/complex.htm (for additional background)

The teacher will show several examples of plotting complex numbers on the number plane, using ordered pairs to identify the numbers:

\[(1, 1), (1, -1), (2, 5), (-2, 5), (3, 4)\]
The teacher will then ask students which number they think is farthest from the origin. This will motivate reintroduction of the Pythagorean theorem in the next section. Some students may believe that (2, 5) is the same distance from the origin as (3, 4) because the coordinates sum to 7 in each case. They can check this with a ruler or piece of yarn.

Finally, the teacher should show students how to write the same complex numbers using $a + bi$ notation:

- $1 + i$
- $1 - i$
- $2 + 5i$
- $-2 + 5i$
- $3 + 4i$
- $3 - 4i$

If students need clarification about how to plot complex numbers in $a + bi$ form, they can watch the following video.

SEE: Complex Numbers, Part 3—The Complex Plane
www.youtube.com/watch?v=EQviqyryrDxA

Practice and Application (time: 15 minutes)
At a center table or posted in front, the teacher will set up a piece of graph chart paper where the scale is 1 block per 1 inch. Students will plot two complex numbers of their own creation on the chart paper. Using a ruler, students should measure how many units each point is from the origin. At this point, the teacher can ask students whether they can think of a way to calculate these distances. Some students may remember the Pythagorean theorem ($a^2 + b^2 = c^2$), and the teacher will reintroduce it and show how it used with complex numbers. So, for example, just as the hypotenuse of a right triangle with legs of 3 and 4 is the square root of $3^2 + 4^2$ ($9 + 16 = 25$, thus 5), the distance from the origin of the complex number (3, 4) or $3 + 4i$ is the square root of $3^2 + 4^2$, which is 5 units from the origin on the complex plane. Students will then show understanding of the concept by finding the number of units from the origin for the two complex numbers they plotted previously.

Review and Assessment (time: 5 minutes)
The teacher will remind students that today they learned about a new kind of number:

Complex numbers that live on a number plane instead of a number line. If complex numbers are really numbers, then we should be able to do arithmetic with them. So, how do we do that?

Students will then write in response to this prompt:

If you were trying to add the complex number $(2, 7)$ or $2 + 7i$ with the complex number $(3, -5)$ or $3 - 5i$, what answer do you think you should get? Explain your reasoning through written expression and/or illustrations.

Extension
Some students may need more extensive review of right triangles and the Pythagorean theorem before applying it to complex numbers. The teacher can provide some hands-on practice by having students calculate and measure the hypotenuses of triangles measured in floor tiles. For example, a triangle 1 tile wide and 2 tiles high will have a hypotenuse of the square root of $1^2 + 2^2$, which is the square root of 5 (roughly 2¼ tiles).
Lesson 2

Adding, Subtracting, and Multiplying Complex Numbers

**Goal**
Students will perform operations on complex expressions and plot them on the number plane.

**Do Now** (time: 5 minutes)
On chart paper, students will plot the complex number $2 + 7i$. The students should then find five other complex numbers that would have the same distance from the origin. In cooperative groups or pairs, students will explain how they know the distance is the same.

**Hook** (time: 10 minutes)
The teacher will ask students to read aloud what they wrote during the previous lesson about how complex numbers should be added. They will then discuss the idea of adding as a shift: for real numbers, adding 12 means move 12 steps to the right on the number line. The teacher should ask:

- Can we still think about the idea of shifting when we add complex numbers?

Students should understand that they can still view adding as shifting, but that the shifting will have two components, horizontal and vertical. The teacher will ask students to simplify the following expression using arithmetic to illustrate this point, then show the operation with right triangles drawn on the number plane:

$$(3 - 4i) + (2 + i) \quad \text{Answer:} \quad 5 - 3i$$

The class can then use diagrams to discuss if adding complex numbers should be commutative, as it is with real numbers. Students can also look at subtraction as the opposite of adding—it’s still a shift but moving in the opposite direction.

**Note:** The teacher may need to conduct a brief review of the commutative and distributive properties during this lesson if students have not used them recently.

The teacher will continue as follows:

Adding and subtracting complex numbers is simple enough, but multiplying is much harder to wrap your head around. Sometimes we think of multiplication in terms of groups, where 6 times 8 means you have 6 groups of 8. But what would it mean to have (2, 1) groups of (4, -2)? That doesn’t make sense. However, when complex numbers are written in a + bi form, we can use standard algebraic operations to multiply them.

The teacher will then ask students to simplify the following expression and explain the steps they use to arrive at the answer:

$$(2 + i)(6 - 2i) \quad \text{Answer:} \quad 12 - 4i + 6i - 2i^2, \text{ which simplifies to } 14 + 2i \text{ because } i^2 = -1$$

**Note:** Some students may forget to convert the $i^2$ to -1. This is a good time to emphasize that $i$ is not a variable but a number.
The teacher will continue as follows:

So multiplying complex numbers is possible, but what does it mean?

We can also think of multiplication in terms of scaling (stretching and shrinking). When you multiply a real number by 2, you double its distance from zero. With complex numbers, you will still see stretching and shrinking, but you will also see another motion: turning. During the lesson we will look at how complex numbers move when you do operations on them.

**Presentation** (time: 15 minutes)
The teacher will return to the BetterExplained website used in Lesson 1 and play the video from 5:00 to 13:15. This portion of the video includes “Finding Patterns,” which introduces the idea that multiplying by \( i \) produces a counterclockwise rotation around the number plane. It also includes the section “Understanding Complex Numbers,” which reviews the use of the Pythagorean theorem to compute the distance of complex numbers from the origin and goes on to give a real-world example of how multiplying complex numbers produces rotation. After the video, the teacher should project the “A Real Example: Rotations” section of the website to work through the calculations and plotting with students.

**SEE:** A Visual, Intuitive Guide to Imaginary Numbers
betterexplained.com/articles/a-visual-intuitive-guide-to-imaginary-numbers

**Practice and Application** (time: 15 minutes)
Students will practice operations with complex numbers by doing the following exercises. Students will have the option to complete the problems set independently or with a partner. It is the responsibility of the teacher to understand the learning needs of the student population and to modify the complexity of the questions accordingly. The teacher will need to differentiate and scaffold the learning based on their understanding of the student population. Outlining of the problems or sentence starters may need to be provided to students that need additional support.

1. Suppose \( Z_1 = 5 + 2i \) and \( Z_2 = 3 - i \)
   a. On the complex plane, plot the location of \( Z_1 \) and \( Z_2 \).
   b. Then multiply \( Z_1 \) with \( Z_2 \) to get \( Z_3 \).
   c. Plot the location of \( Z_3 \) on the same sheet of graph paper.
   d. Calculate the magnitude of \( Z_1 \), \( Z_2 \), and \( Z_3 \).
2. This multiplication comes out to equal a real number: \((6 - 3i)(2 + i)\). Why does that happen?
3. Suppose \( Z = 3 + 4i \). Show how to compute \( Z^2 - 6Z \).

**Review and Assessment** (time: 10 minutes)
In problems 2 and 3, students did operations with complex numbers where the result came out to be a real number. In problem 1, the result was a complex number. Students will now design two problems of their own: one where the result is a real number and one where the result is a complex number. When finished, students in pairs can swap questions and try to answer them—tutoring each other as they walk through their problems.

**Extension**
Students can look for more numbers that satisfy the condition of the “Do Now” problem (where they look
for numbers with the same magnitude as $2 + 7i$) Some students may believe that there are only 8 such numbers because they are using triangles with legs of 2 and 7 in each quadrant of the plane. Give them a piece of yarn to use to look for more numbers with the same distance. They should eventually realize there is an infinite set of solutions, all located along a circle with radius of the square root of 53.

**Lesson 3**

**Powers of Complex Numbers**

**Goal**

Students will describe the spiral pattern created by successive powers of a complex number. They will also make sense out of simplifying radicals by looking at the magnitude of each complex number in the sequence.

**Do Now** (time: 10 minutes)

Students will plot a complex number, multiply it by $1 + i$, plot the result, and compare distances.

**Hook** (time: 5 minutes)

The teacher should show students the images below and ask them how these images might be related to complex numbers. A “Complex Number Designs” Activity Sheet with larger versions of these images and without the notation is located on p. 6.10.1 in the Supplement.

Note: Each image was made by taking a complex number and raising it to the second power, third power, fourth power, and so on. The appearance of the image depends on the angle and the magnitude of the original complex number. If the magnitude were exactly 1, the image would just be a simple circle with radius 1. The greater the magnitude, the looser the coil of the spiral. The greater the angle, the more space there is between successive dots.
Presentation (time: 5 minutes)

Students will present their work from the Do Now problem. Before proceeding to the next activity, the teacher will ensure that all students are multiplying the complex numbers and plotting them correctly on the complex plane.

Practice and Application (time: 35 minutes)

Students will construct a sequence of complex numbers based on the powers of \((1 + i)\). Term 1 in the sequence is \((1 + i)\). They will multiply \((1 + i)(1 + i)\) to get the next term in the sequence, \(2i\). They will take that result and multiply it by \((1 + i)\) to get the next term in the sequence, \(-2 + 2i\). For each power of \((1 + i)\), students should show the multiplication they used to get the result. Students will also plot each complex number on the complex plane. After doing this up to \((1 + i)^7\), students should predict where they believe \((1 + i)^8\) and \((1 + i)^9\) would be on the plane. Sentence starters will be provided to the population of students that need assistance in breaking down sentences.

After plotting the powers, students will describe the patterns they see and talk about how they made their prediction for \((1 + i)^8\). Each time the power goes up by one, what happens to the location of the complex number? Students should be able to talk about both angle and magnitude. As students use the Pythagorean theorem to calculate distance, they should be able to give both a rational approximation of the distance and to state the distance in radical form.

The calculation of distance will raise the issue of simplifying radicals. Students will be able to see physically on the graph paper that the square root of 8 is twice as long as the square root of 2, i.e., it takes two root-twos to make a root-eight, which can be expressed as \(2\sqrt{2}=\sqrt{8}\). This will fit in with a pattern that students probably have observed in the powers of \((1 + i)\). In the sequence of powers, the magnitude of the complex number doubles every other time. \((1 + i)^2\) has a magnitude of 2, while \((1 + i)^4\) has a magnitude of 4.

While discussing patterns, students should eventually arrive at this conclusion:

Each multiplication results in rotating by 45 degrees and stretching by a factor of \(\sqrt{2}\).

After a thorough exploration of the powers of \((1 + i)\), the teacher should ask students to speculate about the powers of another complex number, such as \((-1 + i)\) or \((1 + 2i)\).

Will it behave in basically the same way? Discuss what will be similar and what will be different.

Then the teacher will show the following video “Powers of Complex Numbers” (twice). In the video, students should notice that some complex numbers have greater rotation than others. This depends on the angle of the original number. Similarly, some complex numbers have greater stretching than others. This depends on the magnitude of the original number. The teacher will provide students with a two-column notes graphic organizer with this guiding question:

What happens to each complex number when it is raised to a higher power? Why?

Students should diagram the original complex number in the left-hand column and explain or show what happens to it when raised to a higher power in the right-hand column.

SEE: Powers of Complex Numbers
www.youtube.com/watch?v=86FmwaaDH60
Review and Assessment (time: 5 minutes)

Students will write in response to this prompt:

We saw today that multiplying by \((1 + i)\) increases the magnitude of a complex number by a factor of 2.

Do you think that multiplication will always make the magnitude get bigger? Or would it be possible to multiply a complex number by another complex number and end up with a new complex number that is closer to the origin? Explain your thinking.

Extension

Students will see that increasing powers of \((1 + i)\) forms a spiral, as the location of the complex number twists 45 degrees and gets farther from the origin each time. They might then think about following the spiral in the opposite direction. If they start at \((1 + i)^5\), for example, and go backwards to \((1 + i)^4\), they will see that each time the exponent decreases by one, the number rotates clockwise and shrinks in magnitude. They should follow this sequence to speculate about how to plot \((1 + i)^0\), \((1 + i)^{-1}\) and so forth. After that, they could speculate about fractional powers, e.g., where would \((1 + i)^{\frac{1}{2}}\) be located on the complex plane?

Lesson 4

Julia Sets (2 Days)

Goal

Students will recognize that mathematics can help us to appreciate beauty.

Lesson 4—DAY 1:

Do Now (time: 10 minutes)

Students will plot the complex number \(2 + 3i\) on the complex plane and label it \(Z_1\). Then they will do the arithmetic to multiply \(Z_1\) by itself. To that result, they will add another complex number, \(1 + 5i\). The final result will be called \(Z_2\) and should be plotted on the complex plane (using the same piece of graph paper).

Hook (time: 10 minutes)

The teacher will project some color pictures of Julia Set fractals and tell students that these images are generated by a computer using complex numbers. Today they will learn how they are made. The arithmetic is not complicated; it’s actually the same arithmetic they just did in the Do Now—squaring and adding. The computer can do the arithmetic a lot faster, of course, and it doesn’t mind doing the same task over and over again.

SEE: Julia Set Posters
www.zazzle.co.uk/julia-set-posters

Students might enjoy the following video, which shows the infinite detail of a Julia Set, as it zooms in repeatedly:

SEE: Julia Set Fractal Zoom
www.youtube.com/watch?v=gruJ0S3TTtI
Or this one, which shows an animation based on changing the value of the constant that makes the Julia set:

**SEE:** Animated Julia Set
www.youtube.com/watch?v=w-nIYBjEkrk

**Presentation** (time: 10 minutes)
The teacher will continue the lesson using the script below. As each of the italicized vocabulary words is introduced, the teacher (or better, a student) should write the term and a brief definition on a card to be posted on a word wall in the classroom.

The teacher will say:

Take a look at the graph paper where you plotted the two complex numbers. Which number is closer to the origin, $Z_1$ or $Z_2$? What do you think would happen if you took $Z_2$ and did the same operation (squaring it and then adding $1 + 5i$) to get $Z_3$? Where might $Z_3$ be on the complex plane? How about $Z_{10}$?

Students will observe that the numbers get to be far from the origin very quickly and won’t even fit on the paper.

The teacher will explain:

When the number gets too big we say it escaped. We count how many times it takes for the number to escape. Did it escape at $Z_{14}$? Or was it faster and it escaped at $Z_6$? It all depends on which $Z_1$ you start with. Mathematicians call $Z_1$ the seed because it’s the number you start with. Then each time you do the process of squaring and adding, it’s called an iteration. So we set up a color scheme and assign one color, maybe yellow, to all the seeds that escape at 14 iterations. The seeds that escape at 12 would be green, perhaps. And that is how the picture is made, just by putting a little dot of color on each seed according to its rate of escape.

**Note:** Technically, the seed doesn’t escape; the sequence based on the seed escapes, but that distinction is not important here.

The teacher will continue:

Of course, it’s possible that some of the seeds won’t escape. Maybe we run iterations 1,000 times, and the $Z$ number is still on the paper. These seeds are trapped, and they get their own special color. Often it’s black or dark blue. We can look for that on some of the posters. Where do the trapped seeds tend to be located?

Students can then watch a 10-minute video about how Julia Set images are made:

**SEE:** How Julia Set Images Are Generated
www.youtube.com/watch?v=2AZYZ-L8m9Q

**Note:** This video mentions, but doesn’t emphasize, coloring according to rates of escape. It focuses mostly on using one color for trapped seeds and another color for escaping seeds.

On the board in the front of the room, the teacher should set up a chart paper graph with the axes centered in the middle of the paper, using a scale of .1 for each block of the chart paper. Each student should have a piece of graph paper, also set up with a scale of .1 for each block. The size of the paper will
be used to determine escape. If $Z_5$ goes off the page, then the seed escaped at iteration 5, and it is given a certain color. Students should agree in advance on the color scheme, including a special color for the trapped seeds.

The calculations are very difficult to do by hand, even with an online calculator like the ones below.

**SEE:** Complex Number Calculator
www.mathsisfun.com/numbers/complex-number-calculator.html

**SEE:** Wolfram Alpha
www.wolframalpha.com/

The teacher should project the complex number calculator and do the operations while the students are in charge of plotting the sequence of $Z$ values and noticing when the sequence has escaped (or if it is trapped). On the board, the teacher should list the different seeds the class will be testing, such as:

- $Z_1 = .8 + .5i$
- $Z_1 = -.5 + .3i$

Students should call out which seed to try next. With the complex number calculator, the teacher will take that seed, square it and then add the complex number $.4 + .6i$. In the formula $f(z) = z^2 + c$, it’s necessary to pick a value for the constant $c$. For this experiment, $c$ is $.4 + .6i$. This equation shows how to get $Z_2$ from $Z_1$:

$$Z_2 = Z_1^2 + .4 + .6i$$

Students will plot $Z_2$ on the graph paper at their seats and label it as $Z_2$.

The next step is to use $Z_2$ as a way to get $Z_3$:

$$Z_3 = Z_2^2 + .4 + .6i$$

Note: Copying and pasting helps a lot when using the online calculator, to avoid having to type the same number in again. Don’t forget to put parentheses around the complex number that is being squared. The online calculator will have some round-off error, so if students see a number like 3.0000000000000004, they should just call it 3 when they are plotting it.

And then $Z_4$ comes from $Z_3$, and so on. The class is making a sequence of $Z$ values and trying to decide if the sequence is trapped or if it will escape off the paper. The class should discuss how long they want to wait before deciding if a seed is trapped. (It might be 20 iterations for the purpose of this experiment.)

As the class determines the escape rate for each seed, a student should come up to the chart paper and make a colored dot. Note that they are coloring the seed, the original $Z_1$ value they started with (not the place where it escaped). If they have decided that the sequence is trapped, they should use the color that was selected for trapped seeds.

As an Exit Ticket for DAY 1 of the lesson, students will respond to this prompt:

Why do you think this branch of mathematics has developed only since the advent of computers?

(The students should see that the calculation process is very tedious to do by hand.)
Lesson 4—DAY 2:

Do Now (time: 5 minutes)

Students will explain the difference between “trapped” and “escaped” seeds in a Julia Set.

Practice and Application (time: 30 minutes)

Next, students will go to the Julia Set Generator and enter the values .4 and .6 where it says “Julia original coordinates.” This sets the value of the constant \( c \) in the formula \( f(z) = z^2 + c \). The teacher will explain:

- We used the complex number .4 + .6\( i \) for our constant \( c \), but we could have picked something else.
- When you change the value of \( c \), you get a different Julia Set.

Students can now play with the online tool to try out different values of \( c \). They can also experiment with the coloring. Their goal is to make an image they find aesthetically pleasing.

SEE: Julia Set Generator

www.easyfractalgenerator.com/julia-set-generator.aspx

Review and Assessment (time: 20 minutes)

If possible, the teacher should arrange for a visitor to attend this part of the class. Students should print out their favorite Julia set images and show them to the class and the visitor. Students will then explain orally how the formula \( f(z) = z^2 + c \) is used to create the image. Their goal is to explain the process clearly enough so that the visitor can explain it back to them. Students will be offered a procedural graphic organizer to support them in scripting the steps and using the appropriate vocabulary prior to presenting.

Extension

Julia Set images are made with the expression \( z^2 + c \). It’s possible to make other images using a different expression. Students can explore this on the internet to see what the images look like when the expression is changed.

Lesson 5

Using the Quadratic Formula to Solve Equations with Complex Roots

Goal

Students will apply the quadratic formula to solve equations with complex roots.

Do Now (time: 5 minutes)

Students will complete the following problem independently:

Suppose \( Z = 2 + 5i \). Show how to compute \( Z^2 - 4Z \).

Hook (time: 10 minutes)

The teacher should follow up on the Do Now, explaining as follows:

- We just saw that if \( Z = 2 + 5i \), \( Z^2 - 4Z \) will come out to be -29. So that’s an example where we do some operations on a complex number and come up with a real number as the final result. Let’s see why that happens on the complex plane.
On the board the teacher should next show the physical steps of squaring $2 + 5i$ and landing at the point $-21, 20$. Then the teacher should use right triangles with legs of 2 and 5 to show that four of those triangles will move the graph to $-29$ on the real axis, as in the diagram above.

**Note:** When subtracting $4Z$, move down 5 and left 2. Subtraction is a shifting motion, like adding, only it goes in the opposite direction.

**Presentation** (time: 10 minutes)

Then the teacher should say, the fact that $2^2 - 4Z$ comes out to be $-29$ could come in handy.

Suppose someone asked us to solve this equation:

$$x^2 - 4x + 29 = 0$$

Let’s do a quick graph on Desmos to see if we could find a solution for that.

**SEE:** Desmos Graphing Calculator

www.desmos.com/calculator

We will graph this and see where the graph touches the x axis:

$$f(x) = x^2 - 4x + 29$$

Hey, where’s the graph? (It won’t show using Desmos’ default scale.)

Students should realize that $f(0) = 29$, so the graph is way up on the screen. The teacher should scroll up to verify that.
Then the teacher will continue:

How would you find \( f(x) \) such that \( f(x) = 0 \)?

It seems like it’s impossible. But of course the Desmos graph is looking only in the world of real numbers. This is not a picture of the complex plane. We see now that there are no real roots for this function. And we might just stop there and say it can’t be done.

The teacher should remind students of the introductory lesson about complex numbers—the idea that when mathematicians run into this kind of situation, they just make up a new kind of number. So that quadratic expression does have roots; it’s just that they’re not roots found on the real number line. Instead, they have to look for a two-dimensional number on the complex plane. The quadratic formula will help find them. The coefficients are identified as follows: \( a = 1, \ b = -4, \ c = 29 \). The teacher should plug those into the quadratic formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = \frac{4 \pm \sqrt{-100}}{2}
\]

Students should notice that there is a negative number under the square root sign. This expression can be written like this:

\[
x = \frac{4 \pm \sqrt{100 \cdot -1}}{2}
\]

It simplifies to this:

\[
x = \frac{4 \pm 10\sqrt{-1}}{2}
\]

Because the square root of negative 1 is \( i \), the expression can be rewritten again as follows:

\[
x = \frac{4 \pm 10i}{2}
\]

Doing the division by 2 results in this:

\[
x = 2 \pm 5i
\]

The teacher should say:

So there it is: \( 2 + 5i \) is the number we started out with. We saw physically on the complex plane that if you square that complex number and then subtract four of them, you land on -29. Add 29 to that and you really do get zero! And the quadratic formula tells us that there is another root that will also work: \( 2 - 5i \).
Practice and Application (time: 20 minutes)
The teacher will write the quadratic formula on the board and give each student a different quadratic equation in standard form (see examples below), with \( f(x) = 0 \). The examples should include a mixture of positive and negative coefficients. Some of the equations should have one real root, some two real roots, and some no real roots. Students should practice using the quadratic formula with their coefficients. The students whose problems have only one real root will notice that the discriminant (the expression under the radical) evaluates to zero. The teacher should discuss the role of the discriminant and what it tells about the roots.

\[
\begin{align*}
x^2 + 4x + 4 &= 0 & \text{Solution: } & (-2) \\
-4x^2 + 4x + 4 &= 0 & \text{Solutions: } & (1/2 - \sqrt{5}/2), (1/2 - \sqrt{5}/2) \\
2x^2 + 4x + 4 &= 0 & \text{Solutions: } & (-1 + i), (-1 - i) \\
-4x^2 - 4x - 4 &= 0 & \text{Solutions: } & (-1/2 + 2i\sqrt{3}), (-1/2 + 2i\sqrt{3}) 
\end{align*}
\]

The teacher should also present some quadratic equations where students need to do some transformations in order to have the form \( f(x) = 0 \), such as the following:

\[
\begin{align*}
x^2 + 2x &= -2 & \text{Solutions: } & (-1 + i), (-1 - i) \\
-x^2 - 2x &= 2 & \text{Solutions: } & (-1 + i), (-1 + i)
\end{align*}
\]

Review and Assessment (time: 10 minutes)
Students will write out all the steps for one example and explain the solution to the class.

Lesson 6
Finding Roots in the Complex Plane—Part 1

Goal
Students will use the geometry of the complex plane to estimate values of complex roots for higher degree polynomials.

Do Now (time: 5 minutes)
Students will show how to find the roots for this quadratic equation:

\[2x^2 - 4x + 5 = 0\]

Hook (time: 5 minutes)
Using Desmos, the teacher will project a graph of \( f(x) = x^3 + 8 \) and ask students where \( f(x) = 0 \). They should observe that the graph has a zero at \( x = -2 \).

The teacher should say:

So \(-2\) is a root for the equation \( x^3 + 8 = 0 \). And we know if we cube \(-2\) we get \(-8\), so add \(8\) to that and you get \(0\). Okay, that’s a root. But are there any other roots for that equation? It’s a cubic, right? We have the possibility of three different roots. Where are they?
Then the teacher should remind students that the Desmos graph is drawn using real numbers and only shows real roots. If there are any other roots for the equation, they would have to be complex numbers. The teacher should explain that the method for finding those complex roots will be what students learn today and tomorrow.

**Presentation** (time: 15 minutes)

The teacher should show the video, “Powers of Complex Numbers” again to remind students of what happens when complex numbers are raised to a power:

**SEE:** Powers of Complex Numbers
www.youtube.com/watch?v=86FmwaaDH60

Before proceeding to the rest of the lesson, the teacher should check to see that students are able to articulate that the amount of rotation depends on the angle of the original number, while the amount of stretching depends on the magnitude of the original number. Students will turn and talk with their table partners to explain their understanding. The teacher will circulate to assess the students’ understanding of the concept.

The teacher will post a piece of 1-inch chart paper, with axes marked, and put a large dot at the location (-8, 0) and tell students that this is their target—they need a number that will equal -8 when cubed. Each student should come up to the chart paper and mark a dot for a complex number that might work as a cube root of -8. They should justify their choice to the class. Then the teacher will project Wolfram Alpha and test each student’s proposed root.

**SEE:** Wolfram Alpha
www.wolframalpha.com

For example, if a student suggests \((1 + 2i)\) as a possible solution, the teacher should put this expression into Wolfram Alpha: \((1 + 2i)^3\). Wolfram Alpha gives the result numerically and also shows a picture of the complex plane. Students will observe that the cube of \(1 + 2i\) lands in Quadrant 3, but it is very close to the x-axis. So the angle is not too bad. But the goal is to hit the x-axis at -8, 0. Wolfram Alpha will also calculate the magnitude (given as the radius in polar coordinates). The magnitude is a little over 11, again not too bad.

When students make their original guesses, some will probably suggest roots that are not in the correct quadrant. It’s important that the teacher go ahead and test these guesses on Wolfram Alpha, so that students can see where the result lands when the number is cubed. After trying a few guesses, the teacher should ask students what they know about the angle and the magnitude of the root they are looking for. Students should be able to realize that the angle must be 60 degrees if three rotations are going to make a 180 degree turn, and the magnitude must be 2. At this point, the emphasis should be on turning and stretching.

**Practice and Application** (time: 15 minutes)

Students will now work individually at computers and use Wolfram Alpha to fine-tune their approximations of a root for \(x^3 = -8\). The class should agree on how close the result needs to be to the location (-8, 0). If a student happens to remember from geometry class that 60 degrees makes a special right triangle, it’s worth pursuing the implications of that. But if it doesn’t come up, it’s also okay for students simply to approximate the location of the root.
Review and Assessment (time: 15 minutes)

Students will use 1-inch chart paper to make a poster to show a solution for the equation $x^3 = -8$. Using a protractor to measure a 60 degree angle and a ruler to measure a distance of 2”, students should mark the location of the root in Quadrant 1. They will also indicate where $x^2$ would be and finally where $x^3$ would be. The scaled diagram should demonstrate that students understand the rotation and stretching that occur as $x$ is raised to a power. (They do not need to show the other root in Quadrant 4, although some students may think of that on their own.)

Extension

Students can look for other roots in the complex plane. For example, what if a student wanted to have $x^5 = -8$ instead of $x^3 = -8$?

- How many roots would s/he expect to find?
- Where would they be located?

Students can sketch a diagram on chart paper to make a guess and then use Wolfram Alpha (see p. 6.8.23) to test out their guesses.

Lesson 7

Finding Roots in the Complex Plane—Part 2

Goal

Students will apply algebra skills to find exact values of complex roots for higher-degree polynomials.

Do Now (time: 5 minutes)

Given the polynomial division problem $x^3 + 3x^2 + 4x + 2 / (x + 1)$, students will use a multiplication diagram (area model) to show how to find the other factor (this skill was developed in Unit 4, Polynomials). A sample diagram is shown below.

<table>
<thead>
<tr>
<th>$x^2$</th>
<th>2x</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>$x^3$</td>
<td>2x²</td>
</tr>
<tr>
<td>1</td>
<td>$x^2$</td>
<td>2x</td>
</tr>
</tbody>
</table>

Hook (time: 5 minutes)

The class will view the posters from the previous lesson, which show the geometry of the equation $x^3 = -8$. The teacher will remind students that they should expect three roots for this cubic equation—where are the others? Students should be able to state that there is a root at -2, 0 (the real root). They may also speculate that the third root is located in Quadrant 3 or 4, as a reflection of the root in Quadrant 1. If students object to searching for the third root by guess-and-check, the teacher should tell them that the point of today’s lesson is to learn how to find these roots more directly by using algebra.
Presentation (time: 20 minutes)

Then teacher should project Wolfram Alpha and type in the equation $x^3 = -8$. Wolfram Alpha will then solve the equation and display the roots plotted on the complex plane. Using the projected diagram, the teacher should mark a right triangle with legs of 1 and $\sqrt{3}$. Students should then use the Pythagorean theorem to find the hypotenuse of the right triangle. This will allow them to verify that the root’s magnitude is indeed 2, so when it is cubed the magnitude will be 8. This is a good time to emphasize the importance of irrational numbers: $\sqrt{3}$ is a number that will make exactly 3 when squared.

See: Wolfram Alpha
www.wolframalpha.com

Students should compare today’s projected diagram to the posters that they made in the previous class and see that their posters show a root in the same place that Wolfram Alpha shows. Wolfram Alpha also shows another root in Quadrant 4. Students may notice that the root in Quadrant 4 appears to be a reflection of the root in Quadrant 1. They can come up to the board and draw a right triangle in Quadrant 4 which will be congruent to the triangle in Quadrant 1. The class can then discuss that the magnitude of the Quadrant 4 root must be the same as the magnitude of the Quadrant 1 root, so both of them can be cubed to make a magnitude of 8.

Some students may have noticed the circle that Wolfram Alpha draws through all three roots. The teacher can ask students what the radius of the circle is. (It’s 2.) The teacher can next put a dot somewhere along that circle and ask:

Would that complex number also be a root for the equation? After all, it has a magnitude of 2, so we can cube it to make a magnitude of 8. Does that mean that it works as a root?

This will bring the discussion back to the idea of rotation. Multiplication of complex numbers involves both stretching and turning. It’s not enough for a root to have the correct magnitude. It also needs to have the correct angle. In this case, that angle is 60 degrees, as students demonstrated on their posters from the previous day.

Now that students have seen the roots displayed on Wolfram Alpha, the teacher should ask this question:

How do we find those roots ourselves? We can see that the complex number located at 1,$\sqrt{3}$ works out perfectly, but how would we ever arrive at that? And how would we know that there are additional roots in other quadrants?

If students have completed a Performance Task for the unit on polynomials, they should now take a look at that work and remind themselves of what they know about finding roots for higher-degree polynomials. At that time students were looking only at real roots, but the same principles will apply. They can then apply the remainder theorem to realize this:

If $x = -2$ is a root for the equation $x^3 + 8 = 0$, then $(x + 2)$ must be a factor of $x^3 + 8$.

Students can use polynomial division to find the other factor. They are looking for an expression to put in the blank:

$$(x + 2) \cdot (\quad \quad \quad ) = x^3 + 8$$

The class should work together at the board to make a multiplication diagram similar to the one from
the Do Now problem. It should look something like this when it is filled in:

<table>
<thead>
<tr>
<th></th>
<th>x^2</th>
<th>-2x</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>x^3</td>
<td>-2x^2</td>
<td>4x</td>
</tr>
<tr>
<td>2</td>
<td>2x^2</td>
<td>-4x</td>
<td>8</td>
</tr>
</tbody>
</table>

The diagram will reveal that the other factor is \( x^2 - 2x + 4 \). The teacher should remind students of the zero product property. They are trying to make \( x^3 + 8 = 0 \). They can see that \( x^3 + 8 \) is a product of \((x+2)\) and \((x^2-2x+4)\). So either \((x+2)\) must be zero or \((x^2-2x+4)\) must be zero.

Students know how to make the expression \((x+2)\) equal zero—just use -2 for \(x\). But how can they make \(x^2 - 2x + 4\) come out to be zero? Students may not see immediately that they can write an equation for this. This is a good time to reinforce the power of the equal sign as a constraint. When they write \(x^2 - 2x + 4 = 0\), they are forcing the output of the expression to be zero. They are saying: find an \(x\) that will make this be true. Once it’s written as an equation, students should see that it’s a quadratic equation, and they know how to solve it. They may wish to refer to previous notes on how to use the quadratic formula.

\[
x = \frac{2 \pm \sqrt{4 - 16}}{2} \\
x = \frac{2 \pm \sqrt{-12}}{2} \\
x = \frac{2 \pm \sqrt{4 \cdot 3 \cdot 1}}{2} \\
x = \frac{2 \pm 2 \sqrt{3} \cdot i}{2}
\]

After students divide by 2, they will see that they really do get the same results that Wolfram Alpha got. There is a root at \(1, \sqrt{3}\) on the complex plane and a reflected root at \(1, -\sqrt{3}\).

**Practice and Application** (time: 10 minutes)

With paper and pencil, students should find all three roots for this equation: \(x^3 + 2x + 3 = 0\). From the rational root theorem, they know that potential roots are 1, 3, -1 and -3. When they realize that -1 works as a root, they will know that \((x + 1)\) must be a factor of \(x^3 + 2x + 3\). They can use a multiplication diagram to find that the other factor is \(x^2 - 1x + 3\). They can then use the quadratic formula to find the other two roots.

**Review and Assessment** (time: 15 minutes)

Students will make up three different polynomial equations:
- A cubic equation that has three real roots
- A cubic equation that has one real root and two complex roots
- A quartic equation that has two real roots and two complex roots
For each equation, students will explain orally how they designed it and how they know it works. The teacher will provide the option of peer collaboration if feasible.

**Extension**

Since there is a quadratic formula that works to solve all quadratic equations, students may be curious to know if there are formulas for higher degree polynomials. Students can research this on the internet. While formulas (really complicated formulas!) do exist for third and fourth degree polynomial equations, it has been shown that no such general formula can exist for polynomials with degrees greater than four. It may surprise students to learn that there are some equations that can only be solved by guess and check.

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**CULMINATING LESSON**

*Includes the Performance Task, i.e., Summative Assessment—measuring the achievement of learning objectives*

### Lesson 8

**Complex Numbers for Beginners (2-3 days)**

**Goal**

Students will apply their understanding of complex numbers to a multi-step task that includes demonstrating algebraic and geometric operations, graphing on the complex plane, and explaining the purpose and uses of complex numbers.

**Performance Task**

Students will create a presentation on complex numbers according to the GRASPS scenario below:

- **Goal:** To create a proposal for a “Complex Numbers for Beginners” book that explains the concept of complex numbers, why people need to understand them, and how to perform operations with them and represent them graphically.
- **Role:** The student is a mathematician pitching a new title for the “Beginners” series.
- **Audience:** The editorial board of Wiley Company, publishers of the “Dummies” books.
- **Situation:** Wiley Company is looking for new titles for its mathematics series but board members must be convinced that a book that includes “imaginary” numbers would sell.
- **Product:** PowerPoint slides, a Prezi, or poster board plus an oral presentation outlining the author’s proposal for the new book, including explanation of what complex numbers and how they can be represented on the complex plane, how they are used in a variety of arithmetical and algebraic operations, what practical and artistic applications they have, and why there is a need for a “beginners book” about them.
- **Standards:** The visual and oral presentation must include the following, presented in an organization and style appropriate to the audience and purpose:
  - An explanation of why a “Complex Numbers for Beginners” book is needed, including the importance of the concept in math and its uses in other fields.
  - An overview of the contents of the proposed book, including illustrations.
A definition of complex numbers, including an explanation of \(i\) such that \(i^2 = -1\), and an explanation that complex numbers have the form \(a + bi\) with \(a\) and \(b\) real.

An illustration of how complex numbers are graphed on the complex plane and how the Pythagorean theorem can be used to calculate their magnitude.

A demonstration of how complex numbers are added, subtracted, and multiplied, and how these operations are reflected in movement on the complex plane.

An explanation and demonstration of how to solve quadratic equations with real coefficients that have complex solutions.

Optional Standards:
The following go beyond the targeted standards but do reflect the content of the unit and may be required at the teacher’s discretion:

- An illustration of how complex numbers raised to successively higher powers can be plotted on the complex plane to create spiral patterns.
- An explanation of how Julia Sets are created and what their color coding shows.
- A demonstration of how higher-degree polynomials with complex roots can be solved using the geometry of the complex plane and algebra skills.

While this project can be an independent activity, it may work best if students complete it as a team, in pairs, or as a whole group. If students do work jointly, the teacher should take care that all students have a role in all parts of the task to ensure that they get the full benefit of the learning opportunities in the project and demonstrate that they have met its goals.

Lesson 8—DAY 1:

Do Now (time: 5 minutes)

Students will explain, orally and in writing, the meaning of the expression \(i^2 = -1\) and state why mathematicians created the “imaginary” number \(i\).

Hook (time: 5 minutes)

The teacher will show students an example of a book from the *Dummies* series and explain that they will be creating a proposal for a “Complex Numbers for Beginners” book. The teacher will ask students to consider why the publishers uses “Dummies” in the titles of these books and what that implies about audience and purpose (e.g., not that the readers are stupid but that they are not experts in the field, so the books must be clear and straightforward).

Note: Yang Kuang and Elleyne Case’s *Pre-Calculus for Dummies* (2nd edition, 2012) includes a chapter on complex numbers, but it has a different focus than this performance task and covers topics that students will not yet be familiar with. If students find this book on the internet, they should be cautioned against using the complex numbers chapter as a model or source. Indeed, the companion website, “How to Perform Operations with Complex Numbers” could be used as a negative example, as it covers several operations very quickly and without supporting graphs.

SEE: How to Perform Operations with Complex Numbers
www.dummies.com/how-to/content/how-to-perform-operations-with-complex-numbers.html
**Presentation** (time: 20 minutes)

The teacher will distribute the Performance Task and review the scenario carefully with students, with particular attention to the audience, product/performance, and standards/criteria. The standards/criteria may be presented in the form of a rubric, or, if time is available, the teacher may engage students in developing a rubric from them.

**Note:** The teacher should indicate at this time which of the optional standards/criteria will be required, if any. These options may be used to differentiate the task.

Then the teacher will project the following list of topics from the unit and ask students to explain what they recall about each of them. (Students may use their notebooks and materials, as well as items posted in the room, as aids in this review.) The teacher should help clarify any misconceptions but prompt students to work out their own explanations whenever possible.

If students do not readily recall key ideas from these topics or seem to be confused about several of them, the teacher should add a full review session before moving on to completion of the Performance Task.

- Plotting complex numbers on the number plane and determining their magnitude with the Pythagorean theorem
- Adding, subtracting, and multiplying complex numbers
- Computing and plotting powers of complex numbers*
- Creating Julia set images through multiple iterations of a complex number operation*
- Using the quadratic formula to solve equations with complex roots
- Finding complex roots for higher degree polynomials*

*Optional topic in Performance Task at teacher’s discretion

Each of these topics has a graphing component, and as students review them, the teacher should emphasize connections between equations and expressions and their respective graphs.

**Practice and Application** (time: 25 minutes)

Students will develop plans for their “Complex Numbers for Beginners” proposals, outlining the content of each slide, or section of the poster board. The standards/criteria section of the Performance Task can provide an initial outline for the presentation, but students may wish to modify the order and combine or split certain topics. Students may work in pairs as they develop these plans (and simultaneously review the unit content), but each student should produce his or her own presentation.

Students may use algebra tiles, graph paper, and online tools such as the following to facilitate and check their work as they go through the planning process:

**SEE:** Complex Number Calculator
www.mathsisfun.com/numbers/complex-number-calculator.html
Complex Numbers—Wolfram Alpha
www.wolframalpha.com/input/?i=complex+numbers
Desmos Graphing Calculator
www.desmos.com/calculator
Julia Set Generator
www.easyfractalgenerator.com/julia-set-generator.aspx
At the end of DAY 1, students will submit work in progress along with an Exit Ticket stating what aspects of the project are going well and what they need help with or have questions about.

Lesson 8—DAY 2:

Review and Assessment (time: 55 minutes)
As a Do Now at the start, students will create a “to-do” list for completing their projects as the teacher responds individually to the questions and concerns they raised at the end of Day 1. The students should then convert the outlines made the previous day into finished slides, or a poster board, including graphs as required. Graphs created in Wolfram Alpha, Desmos, and other programs can be included by taking screenshots (Ctrl-Prtsc), then pasting and cropping the images. Students should also practice their presentations.

Finally, students will present their projects and receive questions and feedback.

Note: In keeping with the “Complex Numbers for Beginners” (i.e., non-experts) theme, the audience should ideally include students and staff not currently studying Algebra 2.

Extension
Students with an interest in complex numbers might enjoy watching the video below, which shows transformations using complex numbers ranging from modifying a photo to the Mandelbot set.

SEE: Dimensions—Chapter 6: Complex Numbers in Use
www.youtube.com/watch?v=id5JfrSD1i0

POST–UNIT REFLECTION
On meeting the Learning and Language objectives
Connections to Empower Your Future
UNIT: Complex Numbers

Future Ready Connections

This unit encourages students to activate their Future Ready skills, think creatively and critically, and to utilize math tools such as graph paper, protractors, graphing calculators, and other online math software. Throughout the unit, students are encouraged to experiment, reflect, and try new approaches to solving problems using equations and tools.

This freedom to experiment will give teachers the opportunity to evaluate students on their initiative, self-direction, and accountability for completing tasks, actively engaging in critical thinking, and taking responsibility for their own discoveries. Students have the opportunity to create their own math problems (Lessons 2 and 7) and explain the solution to a problem to their peers (Lesson 5), which requires them to take the initiative to create solvable problems and be accountable for other students' understanding. Teachers should reflect on whether or not youth stay on task without prompting and if they push themselves to thoroughly complete each activity, answer their own questions, and create a detailed final product instead of only addressing the minimum required information or waiting for explanations from the teacher.

Youth have many opportunities to strengthen their communication and listening skills through group discussions, oral reflections, written responses, and the presentation of their Performance Tasks. Both oral and written communication and explanations can be evaluated for clarity and effectiveness. Group discussions and partner work are also appropriate times to assess students' ability to work effectively and respectfully with diverse teams by sharing responsibility for collaborative work and validating individual contributions made by team members.

Teachers are encouraged to use the Future Ready Rubric to evaluate students’ growth and are encouraged to have students self-evaluate their progress using the Future Ready Rubric. Students should reflect on how they demonstrated growth, what they could do to further improve their skills, and how they are transferable to other situations and experiences.

Transfer Goal and Essential Question Connections

One Essential Question for this unit is “Why do mathematicians make up new kinds of numbers?” This Essential Question connects to one of the Transfer Goals for this unit which states that students will articulate that mathematics is a human creation that evolves as needed to address inconsistencies. This Transfer Goal is directly addressed in Lesson 1 as students will reflect on how imaginary numbers were created to solve specific math problems. Teachers should consider expanding on this Transfer Goal by having youth reflect on the process of problem-solving and the need to think “outside the box.” Teachers may ask students to reflect on their own problem-solving process when they come across a new or challenging situation. What do they do first? What if their initial attempt doesn’t work? How do they decide to try something new? Teachers may also wish to brainstorm innovative inventions that have changed our world and were created because we needed new tools to solve existing problems. Examples of this may include: the invention of the calculator because doing math by hand was time consuming and prone to error, the invention of the cell phone because communication was limited by being in a certain place at a certain time. By brainstorming these inventions, teachers can make a connection between the concept of the imaginary number’s invention in order to
solve unique problems to the invention of different tools and processes in order to solve modern day problems. Teachers may choose to focus on the concept of mathematics evolving to meet certain needs in order to have students reflect on their own personal growth and experiences. Students can reflect on how their own behaviors and beliefs have evolved over time based on experience or the need to accomplish a specific goal. If mathematicians can introduce a new number to solve a problem, what can students introduce or change in their lives to solve a problem or reach a goal?

Career Exploration Connections

Lesson 3 includes images created by successive powers of a complex number which show various types of spirals and swirls. Many students will notice that these images look like works of art or other interesting graphics. Teachers can note for students that graphic artists often use computer software that depend on mathematical equations. Teachers can expand on this idea by having youth research careers and industries that may also use complex numbers, spiral patterns, and quadratic equations. Students can brainstorm or do an internet search for additional careers such as electrical engineers, quantum physicists, sales analysts, and economists. Teachers should encourage students to see that knowledge of algebra and specifically complex numbers does play a role outside of the classroom and in many career fields.

PYD/CRP Connections

This unit reflects many aspects of Culturally Responsive Practice and Positive Youth Development by allowing for exploration, experimentation, reflection, and scaffolding so that students are active participants in their learning and are a part of the discovery process. This is especially evident in Lesson 2, Lesson 7, and in the Performance Task because students are responsible for creating questions, providing solutions and explanations, and creating a comprehensive presentation on complex numbers. By encouraging youth to take risks and to be part of the discovery process, teachers are grounding the tasks in the students’ strengths and allowing students to have a voice in the classroom and in their own learning. Teachers are encouraged to further differentiate lessons and engage youth as resources in the classroom by having students work in a combination of settings such as in pairs and in small groups.

“Teachers should encourage students to see that knowledge of algebra and specifically complex numbers does play a role outside of the classroom...”

For Technical Assistance with Empower Your Future connections and lessons, please request support by submitting a Coaching Request ticket using the Coaching Feature on TeachPoint.
Complex Numbers Designs
Lesson 3
Data Analysis and Statistics

TOPIC SEASON | Algebra 2—Trigonometry, Statistics, and Probability

This unit is designed for use in long-term programs.
Sections may be adapted for short-term settings.
Unit Designers: K. Chase, N. Koch, and B. Penniman

Introduction

Statistics are all around us, from commercials that claim that “4 out of 5 dentists recommend” a particular kind of toothpaste, to election polls, to randomized trials that determine whether new drugs are approved by the FDA. As consumers of products and news, all students need a basic understanding of statistics in order to address the credibility of claims made in the media. Further, since we live in an increasingly data-driven world, many professions—from business, to medicine, to education—require the use of statistical studies. In order to be prepared to enter these professions, and to be active participants in society, it is essential that students have a working knowledge of the reasoning processes and methods that are involved in the study of statistics.

The “Data Analysis and Statistics” unit is designed to come last in the Spring Season of Trigonometry, Statistics, and Probability and will take about three weeks to complete. Since this unit addresses engaging topics that are directly relevant to students’ lives, it is the perfect unit to teach at the end of the year so that students remain motivated to learn. Options have been provided to reduce this unit to a seven-day study by shortening the Performance Task and not requiring students to perform statistical tests, since they are not clearly required by the standards addressed in the unit.

The “Data Analysis and Statistics” unit focuses on six standards that are aimed at improving students’ mathematical reasoning.

S-IC.1: Understand statistics as a process for making inferences about population parameters based on a random sample from that population.
S-IC.2: Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation.
S-IC.3: Recognize the purposes of and differences between sample surveys, experiments, and observational studies; explain how randomization relates to each.
S-IC.4: Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for sampling.

“Teachers should model how to solve equations, encourage students to attack formulas one step at a time, and allow students to use digital tools that will perform most of the calculations.”
**S-IC.5:** Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.

**S-IC.6:** Evaluate reports based on data.

While all of these standards may seem daunting to teach in three weeks, it is important for teachers to keep in mind that at the core of these standards is the goal that we will increase our students’ mathematical reasoning when we ask them to test hypotheses, draw inferences, and determine that conclusions are valid.

In order to engage students with the standards and content of the unit, teachers should focus discussions around the Essential Questions.

1. How can I collect, organize, interpret, and display data to investigate a question?
2. Why is it important to interpret data carefully?
3. Could the result be due to chance, or is something else going on?

In order to engage in these questions, students will design, conduct, and publish their own statistical studies based on questions of interest to them. No matter what the nature of their studies is, it is important that students consider and prove or disprove the null hypothesis:

1. Could the results be due to chance?

In order to be successful in this unit of study, students must be familiar with measures of central tendency (mean, median, mode), qualitative vs. categorical variables, the concepts of normal distribution and standard deviation, and simple binomial probability. They should also be able to read a histogram. If students do not have this prior knowledge and these skills, the teacher can incorporate brief mini-lessons as needed throughout the unit. (See the Algebra 1 Statistics unit for ideas to teach many of these concepts.)

Even with this prerequisite knowledge, students might find parts of this unit challenging. Students may find that the formulas in this unit look complicated, and although they can be tedious, the calculations required are not very difficult. Teachers should model how to solve equations, encourage students to attack formulas one step at a time, and allow students to use digital tools that will perform most of the calculations. Students might have difficulty with the conceptual elements of null hypothesis and randomization since they can be difficult to grasp. They may also have difficulty with the fact that conclusions drawn from statistical analysis can seem counterintuitive. Furthermore, the lack of absolute certainty in the study of statistics can be disconcerting for some students. Teachers must remember that grappling with these challenges is the essence of the unit. The more discussion about these topics that students can have in class, the more they will be prepared to tackle the tasks asked of them now and in the future.

For adaptation ideas for this unit, see p. 6.11.3 on the right
**SPRING SEASON—Data Analysis and Statistics: Long-Term Programs**

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<tr>
<th>MONDAY</th>
<th>TUESDAY</th>
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<tr>
<td><strong>Week 1</strong></td>
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<tr>
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<td>Lesson 2: Random Sampling of a Population</td>
<td>Lesson 3: Study Types</td>
<td>Lesson 4: The Chi-Squared Test</td>
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<td><strong>Week 2</strong></td>
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<td>Lesson 5: Sample Size, Margin of Error, and Confidence Level</td>
<td>Lesson 6: Experimental Studies</td>
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<td>Lesson 7: Evaluating Statistical Studies</td>
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<td><strong>Week 3</strong></td>
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<td>Lesson 8: Reviewing the Types of Statistical Studies</td>
<td>Lesson 9: Designing and Conducting a Study</td>
<td>Lesson 10: Presenting and Publishing a Study</td>
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**Plan 2 (Short)**

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<th>MONDAY</th>
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<td>Lesson 7: Evaluating Statistical Studies</td>
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When used in short-term programs the final week of the unit, comprising an extended Performance Task, is omitted.

In this version, the Formative Assessment in Lesson 7 can serve as the Summative Assessment for the unit. The Formative Assessments in Lessons 4, 5, or 6 can also be used as Summative assessments for students who are not present for the entire unit.
UNIT PLAN  

Long-Term Programs

Data Analysis and Statistics

Designed by: K. Chace, N. Koch, and B. Penniman

Theme or Content Area: Algebra 2—Trigonometry, Statistics, and Probability

Duration: 10 Lessons / 3 Weeks

Emphasized Standards *(High School Level)*

**STATISTICS AND PROBABILITY**

S-IC.1: Understand statistics as a process for making inferences about population parameters based on a random sample from that populations.

S-IC.2: Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation.

S-IC.3: Recognize the purposes of and differences between sample surveys, experiments, and observational studies; explain how randomization relates to each.

S-IC.4: Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.

S-IC.5: Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.

S-IC.6: Evaluate reports based on data.

Essential Questions *(Open-ended questions that lead to deeper thinking and understanding)*

How can I collect, organize, interpret, and display data to investigate a question?

Why is it important to interpret data carefully?

Could the result be due to chance, or is something else going on?

Transfer Goals *(How will students apply their learning to other content and contexts?)*

Students will recognize the use of sampling when reading statistical studies and determine whether valid data collection practices were used.

Students will use data to make informed decisions or predict future outcomes.

Students will consider the role of chance when attempting to explain an observation.
### Samples
Population parameters

Samples are used to represent populations.

#### Random sampling

In a random sample, each member of the population has an equal probability of being picked.

Identify if a method of sampling is random.

#### Null hypothesis

Experiments often seek to determine if there is a relationship between two variables or a difference between two groups. The null hypothesis asserts that there is no such relationship or difference—any appearance of a relationship or a difference is due to chance variation in the data.

State the null hypothesis in the context of a situation.

#### Chi-squared test

We can only reject the null hypothesis if the experimental result is sufficiently different from what might occur due to chance. Tests like chi-squared help us to assess that difference.

Compute chi-squared statistics to determine if we can reject the null hypothesis.

#### Sample survey
Controlled experiment
Observational study

There are many ways to gather sample data.

Recognize which type of sampling method was used and identify any errors in the method.

**Table continues on p. 6.12.3**
### Population mean

- Proportions
- Margin of error,
- Confidence level

The larger the sample size, the lower the margin of error. The confidence level represents the probability that the sample mean is a good estimate of the population mean (within the margin of error, plus or minus).

### Disproportionality, Overrepresentation, Disparity

Statistical analysis can reveal anomalies in populations.

### Randomization simulations

Randomization simulations allow us to estimate a theoretical probability in lieu of computing that probability directly.

### Hypothesis t-test

Randomized experiments can be used to compare two groups. The t-test is used in situations where a quantitative variable is being measured in the two groups, while the chi-squared test is used when the data of interest is a categorical variable.

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<thead>
<tr>
<th>Students should know...</th>
<th>understand...</th>
<th>and be able to...</th>
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<tbody>
<tr>
<td><strong>Population mean</strong></td>
<td>The larger the sample size, the lower the margin of error. The confidence level represents the probability that the sample mean is a good estimate of the population mean (within the margin of error, plus or minus).</td>
<td>Articulate the connection between sample size, margin of error and confidence level. Apply the “plus or minus” idea of margin of error to find the range of estimates for the population mean.</td>
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<tr>
<td><strong>Disproportionality, Overrepresentation, Disparity</strong></td>
<td>Statistical analysis can reveal anomalies in populations.</td>
<td>Use the vocabulary appropriately in a context; recognize the existence of disparity from a set of data.</td>
</tr>
<tr>
<td><strong>Randomization simulations</strong></td>
<td>Randomization simulations allow us to estimate a theoretical probability in lieu of computing that probability directly.</td>
<td>Use software to conduct a simulation; draw conclusions from the results of the simulation.</td>
</tr>
<tr>
<td><strong>Hypothesis t-test</strong></td>
<td>Randomized experiments can be used to compare two groups. The t-test is used in situations where a quantitative variable is being measured in the two groups, while the chi-squared test is used when the data of interest is a categorical variable.</td>
<td>Determine a hypothesis to test and compute a statistic on the results.</td>
</tr>
</tbody>
</table>
**Assessment Evidence**

Quality questions raised and tasks
designed to meet the needs of all learners

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**Performance Task and Summative Assessment** (see pp. 6.12.35-6.12.40)

*Aligning with Massachusetts standards*

Students will design, conduct, and publish their own statistical studies based on questions of interest to them (see Lessons 9-10). Depending on the nature of the questions, the studies could be sample studies, observational studies, or experimental studies. No matter what the nature of study is, it is important that students consider and prove or disprove the null hypothesis:

Could the results be due to chance?

Each study design must include the following aspects: research question, type of study, independent variable, dependent variable, method, statistical test and/or simulation. Students will conduct their studies, carefully collect and sort data and use appropriate statistical technique(s) to determine whether the results are significant. Students will then organize their materials and prepare brief oral reports to the class about their research studies. The reports should include research questions, descriptions and rationales for research designs, and analyses of results, including statistical tests or simulations. After receiving feedback from their peers and the teacher, students will plan and develop chart-paper posters, or written reports detailing their studies, including all of the elements included in the oral reports, but with a more formal presentation, including tables and/or graphs, statistical results, and analysis. The teacher will scaffold writing or composing activities using tools such as concept maps, outlining tools, or graphic organizers. Students may need sentence starters and story webs to complete writing or composing tasks.

**Pre-Assessment** (see pp. 6.12.11-6.12.13)

*Discovering student prior knowledge and experience*

Students will complete a pre-assessment to test knowledge of:

- computing percentages
- using a frequency table to describe relationships between variables
- understanding basic statistics concepts such as mean
- reading a histogram
- interpreting statistical studies (See Lesson 1)

The teacher may wish to have students complete the pre-assessment one page at a time and pause for review and discussion, thus allowing them to gradually reactivate their prior knowledge and skills.
**Formative Assessments** (see pp. 6.12.11-6.12.32)

*Monitoring student progress through the unit*

- **Lessons 1, 2, 7:** Students share their findings with the class by explaining studies, strategies, or results orally and/or in writing.
- **Lessons 2, 6:** Students write Exit Tickets summarizing what they have learned and/or raising questions.
- **Lessons 3, 8:** Students create and/or analyze examples of statistical studies.
- **Lesson 4:** Students analyze study data and create poster presentations or PowerPoint presentations explaining results.
- **Lesson 5:** Students will present information about a poll of interest and explain what can be inferred from the poll results, given the margin of error and the confidence level.
- **Lesson 6:** Students will conduct an experiment and analyze the results using a simulation based on randomized distribution.
Multiple Means of Engagement

This is the *why* of learning, what makes students engage or disengage. Throughout the unit plan, the student will be provided with as many choices in the level of challenge and complexity as possible in order to recruit and sustain engagement. For example, the teacher will encourage and support students in setting their own personal, academic, and behavioral goals. The teacher will use many strategies to guide students, including reminders, guides, rubrics, checklists, and prompts among other things that focus students on self-regulatory goals. Student tasks will be varied, allowing for active participation, exploration, and experimentation. The teacher will provide differentiated models, scaffolds, and feedback, as well as content material that is culturally relevant and responsive to student’s backgrounds. Most important is that teachers design assignments and tasks with authentic outcomes, and that are purposeful and convey meaning to real audiences.

While statistics may seem like a “dry” subject on the surface, it relates closely to topics of inherent student interest, such as opinion polls about popular culture and politics. Throughout the unit there are opportunities for cooperative learning with partners or small groups, and students frequently their present work to each other. The unit also offers hands-on experimentation with and analysis of statistical studies, including M&M color distribution, computer simulations, and paper helicopter design testing. In addition, each student will have the opportunity to design, conduct, and report his/her own statistical study on a topic of interest.

Multiple Means of Representation

This is the *what* of learning. There are many pathways to conveying information to students. Throughout the unit, the teacher will provide information and materials in several modalities such as diagrams, vocabulary cards, and word walls, posters, and charts with formulas for calculations; and models, videos, and audio for text. The teacher will also demonstrate concepts through hands-on activities.

The way information is displayed should vary, including size of text, images, graphs, tables or other visual content. Where possible, written transcripts for videos and auditory content should be provided. Information should be chunked into smaller elements, and complexity of questions can be adjusted based on prior knowledge. The teacher may consider several tools for presenting the information of key concepts to students: PowerPoint presentations, guided notes/graphic organizers, and videos for more visual learners. The unit utilizes a variety of online tools that are useful for visual learners as well. For the Performance Task in which students create their own studies, the teacher should provide checklists and graphic organizers to help the students classify the information and keep on track. Modifications intended to adjust the unit’s learning and language objectives, transfer goals, level of performance and/or content will be necessary for students with mandated specially designed instruction described in their Individualized Education Programs (IEPs).
Multiple Means of Action and Expression

This is the how of learning. In learning activities students will be provided options for demonstrating what they know and can do. Students will have access to assistive technology and use multiple media. For example, students will have access to word processors with grammar checks, word prediction, and spell checkers. Students could complete projects by making PowerPoint presentations, rapping through music videos, or drawing illustrations. In addition, students will have access to calculators. The teacher will scaffold writing or composing activities using tools such as concept maps, outlining tools, or graphic organizers. Students may need sentence starters and story webs to complete writing or composing tasks. The teacher will also break down long-term goals into short-term reachable goals.

Performance Tasks can be differentiated by content, process or product to address various learner profiles. For the creation of the chi-squared table, students could be given the option to do it by hand or using technology such as Word, Excel, or any other programming that allows for table making and data collection. The class presentation of the study can include a poster, PowerPoint, or through some other form of representation. The student could be encouraged to make a video if they prefer to record themselves and play it back for the class rather than to be “put on the spot” on the day of presentations. The unit allows for students to have their own choices in the study they would like to explore based on their own personal interests. The teacher could create checklists, organizers, and other project planning tools to assist the students in self-monitoring their development of the project. Students should be given high- and low-tech options to compose in multiple media such as text, speech, drawing, illustration, comics, storyboards, design, film, music, visual art, sculpture, or video. Students can use graphic organizers, concept mapping with Inspiration, or drawings by hand, checklists, sticky notes, and mnemonic strategies to better understand and demonstrate comprehension of the material. Opportunities for collaboration and whole-class discussion may be provided as needed. Accommodations intended to enhance learning abilities, provide access to the general curriculum, and provide opportunities to demonstrate knowledge and skills on all performance tasks will be necessary for students with applicable Individualized Education Programs (IEPs) and could benefit all learners.

Literacy and Numeracy Across Content Areas

Reading

Students will read a variety of studies and become familiar with surveys, samplings, and populations. This reading will include interpreting data, tables, and graphs. Students will also read and respond to online statistical study scenarios and multi-step instructions for conducting experiments.

Writing

Students will engage in low-stakes writing-to-learn throughout the unit, including a variety of Exit Tickets summarizing learning from lessons. They will also “write” tables and create posters or digital presentations to report and share their learning.
Speaking and Listening

Students will discuss problems and solutions with partners, small groups, and the entire class and address questions other students may ask about their conclusions. They will make formal presentations of their Performance Assessment study results to the teacher and the class, fielding questions and receiving feedback. They will also provide feedback to their peers.

Language

Students will learn and utilize key vocabulary connected to statistical studies, such as sample, population, parameters, null hypothesis, chi-squared test, sample survey, controlled experiment, observational study, population mean, proportion, margin of error, confidence level, disproportionality, overrepresentation, disparity, hypothesis, t-test, randomization, simulation.

Numeracy

Opportunities for learning mathematics skills are found in the use of percentages, means, statistics, graphs, formulas, tables, and simulations. Making inferences and interpretation of data, key elements of this unit, are higher-order skills that increase numeracy.

Resources (in order of appearance by type)

Print


Websites

Lesson 1


http://freakonomics.com/books/#freakonomics.

“Algebra II Performance Based Assessment Practice Test.” PARCC. Pearson Education. 2015. (see p. 6.14.2)

https://gssdataexplorer.norc.org/

Lesson 2


Lesson 3
www.random.org


Lesson 4
www.youtube.com/watch?v=xVJTIz7Z3Ds

Lesson 5
www.nctm.org/Classroom-Resources/Core-Math-Tools/General-Purpose-Tools/

“Right Direction or Wrong Track.” Rasmussen Reports. Rasmussen Reports, LLC. 2016.
www.rasmussenreports.com/public_content/politics/mood_of_america/right_direction_or_wrong_track

www.realclearpolitics.com/epolls/other/president_trump_job_approval-6179.html

“Methodology.” Rasmussen Reports. Rasmussen Reports, LLC. 2016.
www.rasmussenreports.com/public_content/about_us/methodology


www.raosoft.com/samplesize.html


“Right Direction or Wrong Track.” Rasmussen Reports. Rasmussen. 2016.
www.rasmussenreports.com/public_content/politics/mood_of_america/right_direction_or_wrong_track


Lesson 6
NCTM General Purpose Tools (Download)
www.nctm.org/Classroom-Resources/Core-Math-Tools/General-Purpose-Tools/

Lesson 7

www.cbsnews.com/feature/cbs-news-polls/

http://gss.norc.org/

Lesson 8
www.khanacademy.org/math/probability/statistical-studies/types-of-studies/e/types-of-statistical-studies

Lesson 9

“QuickCalcs t-test calculator.” GraphPad Software. GraphPad Software, Inc. 2016.
www.graphpad.com/quickcalcs/ttest1.cfm

Lesson 10
http://magazine.amstat.org/blog/2013/08/01/poster-and-project/

Materials
Scissors, tape, and paper to make paper helicopters

Bags of M&Ms (large and snack size), or data to represent packages of M&Ms.
(Note: Use plain M&Ms in case of peanut allergies.)
PREREQUISITES: Math skills needed for this unit

Data Analysis and Statistics is the third unit in the Spring Season of Algebra 2. The prerequisite math skills summarized below are taught in units that precede this unit (see Scope and Sequence chart). The following skills will be needed for students to successfully complete the this unit.

Students should know:

- Measures of central tendency (mean, median, mode)
- Quantitative vs. categorical variables
- Normal distribution
- Standard deviation
- How to read a histogram
- Simple binomial probability

Outline of Lessons

Introductory, Instructional, and Culminating tasks and activities to support achievement of learning objectives

INTRODUCTORY LESSON

Stimulate interest, assess prior knowledge, connect to new information

Lesson 1

Introduction to Statistical Studies

Goal

Students will reactivate prior knowledge about formulas, statistics, and making predictions.

Do Now (time: 5 minutes)

The teacher will ask students how they might predict the current favorite cartoon of pre-school children or any other phenomenon they would have to guess about. Students will share their ideas in groups or pairs, brainstorm orally or on paper, or by think-pair-share.

Hook (time: 10 minutes)

The teacher will introduce students to the idea that we can make predictions about an entire population based on data we collect from a sample. Sampling and statistical testing are essential components of many kinds of research, including medical research. As an example of the kinds of problems scientists research with statistical studies, the teacher can play Karen Brown’s New England Public Radio broadcast story
about alternative treatments for Post Traumatic Stress Disorder (PTSD). (A transcript is available on the site for students who need additional scaffolding.)

**SEE:** More Vets Get Alternative Treatment For PTSD, But Not Always Evidence-Based.  

Before the class listens to the story, the teacher should post on the board the term *anecdotal evidence*. The teacher will ask students to listen for the term in the story and then discuss the idea of anecdotal evidence after the story plays. The teacher should provide students with guiding questions and a read-write-think or other graphic organizer to take notes and respond to questions.

**Note:** Many students may not be familiar with Post-Traumatic Stress Disorder, so the teacher may need to review with students prior to listening to the story.

Other examples are studies from *Freakonomics* by Steven D. Levitt and Stephen J. Dubner or other data that interests students.

**SEE:** Freakonomics: The Books  
http://freakonomics.com/books/#freakonomics

### Presentation (time: 15 minutes)

The teacher will remind students that they likely have learned several skills and terms related to statistical studies in previous math or science courses. To review these skills and terms and assess students’ familiarity with them, students will complete a pre-assessment on:

- computing percentages
- using a frequency table to describe relationships between variables
- basic statistics concepts such as mean
- reading a histogram

The “Advanced Statistics Pre-Assessment” Activity Sheet is on pp. 6.14.1-6.14.4 (ANSWER KEY p. 6.14.5). The teacher may wish to have students complete the pre-assessment one page at a time and pause for review and discussion, thus allowing them to gradually reactivate their prior knowledge and skills.

### Practice and Application (time: 25 minutes)

In pairs or individually, students will familiarize themselves with study data as a further introduction to this topic. The *GSS Data Explorer* offers a wide selection of data compiled by the University of Chicago.

**SEE:** GSS Data Explorer  
https://gssdataexplorer.norc.org/

On the home page, students can click on “Explore GSS Data,” then, on the next page, enter search criteria. For examples, students could type in the keyword or choose the subject “education” and narrow the search to the years 2000 to the present. The search will produce a number of results, including one with the description “improving nations education system.” Clicking on that study will take students to a page that provides the survey question and the respondents’ answers, by year. Examining one or more studies on this site will show students how data are collected, arranged, and presented. Students can also examine the data and try to draw some inferences from it.
Students should also begin thinking about a study that interests them for the final project/presentation and take notes on their ideas.

The GSS Data Explorer site allows teachers to establish class accounts and create projects for students, if desired. Also provided are examples of how to extract and analyze data.

Teachers may obtain the Instructor’s Starter Guide for the GSS Data Explorer at:

SEE: Instructor’s Starter Guide for the GSS Data Explorer

**Review and Assessment** (time: 5 minutes)

Students should share their findings with the class by explaining a study that interested them: what was asked and what data were presented. For instance, in the example alluded to above, students could say that researchers asked survey respondents “are we spending too much, too little, or about the right amount on improving the nation’s education system?” Students could also explain that the results show that in most years since 2000, more than ten times as many people respond “too little” than “too much.” The teacher may need to provide students with culturally relevant potential examples students can utilize.

**INSTRUCTIONAL LESSONS**

*Build upon background knowledge, make meaning of content, incorporate ongoing Formative Assessments*

**Lesson 2**

Random Sampling of a Population

**Goal**

Students will make inferences about a population based on a random sample from that population (S-IC.1).

**Do Now** (time: 10 minutes)

Students will find the mean, median, and range of the following data set: 4, 1, 5, 4, 7, 6, 1, 5. (Mean = 4.125; Median = 4.5, Range is 6.) The teacher will lead a review of the terms based on the students’ responses. Students will also create a Word Wall.

**Hook** (time: 5 minutes)

Students will brainstorm how they would answer someone from another country who asked, “Who is the most popular rapper/singer in the U.S.?” The teacher will ask them to think about how they could check their accuracy. Among the suggestions students are likely to make will be the idea of conducting a survey, and the teacher can use that response to make a transition to the presentation on the importance of random sampling.

**Presentation** (time: 10 minutes)

The teacher will explain the meaning of words such as sample, parameter, random sampling, etc., and put them in a Word Wall. The teacher will explain what it means when a study refers to a population,
clarifying the need to identify the parameters being studied and the need for random sampling if the study is to be valid. A sample is considered random if each individual in the population has an equal probability of being picked for the sample. The following website has interactive examples that match the standard. The teacher should facilitate a discussion around some of these examples as a class before having the students practice on their own.

**SEE:** Sampling Methods 2

Students can show their thinking through writing, drawing, or mathematical calculations.

**Practice and Application** (time: 20 minutes)
The best way for students to become comfortable is to see and discuss examples of studies. This can be done as in pairs where the students read a passage and discuss the questions, but have the answers and explanations available to them to check their understanding. The following website has examples:

**SEE:** Sample Assignments under “Making Inferences and Justifying Conclusions HSS-IC.A.1”

**Note:** If students access this site on their own, ask them to scroll down gradually so as not to reveal the answers before attempting to figure them out. The teacher may wish to extract and print the problems and answers on separate sheets.

**Review and Assessment** (time: 10 minutes)
Students will share their results from the assignment and write an Exit Ticket listing strategies they developed for determining the populations being studied.

### Lesson 3

**Study Types**

**Goal**
Students will recognize the purposes of and differences between sample surveys, experiments, and observational studies and explain how randomization relates to each (S-IC.3).

**Do Now** (time: 5 minutes)
Students will explain (with reasons) if the following survey results are valid:

A survey of 50 people at the Golden Corral found that the majority of adults’ favorite restaurant is the Golden Corral.

**Hook** (time: 10 minutes)
The teacher will ask students to write down the name of everyone in the room in a random order. Students should then discuss their method of randomization. This may raise a misconception about the meaning of the term “random.” Some people believe that it means “mixed up.” In their attempt to randomize, students may do something like alternating back and forth between sides of the room. The teacher should emphasize to students the importance of probability in the definition of randomness. In
this context, it would mean that each person in the room has an equal chance of being picked to be first
on the list, equal chance to be second on the list, and so on. Given that criterion, the teacher will ask
students to brainstorm some methods for randomizing the names of people in the room. They might
suggest something like putting the names on individual index cards and then shuffling the cards. They
might also want to use a computer in some way. The teacher can project the website www.random.org as
a source of random numbers.

SEE: Random
www.random.org

Presentation (time: 10 minutes)
The teacher will introduce three types of sampling methods: sample surveys, experiments, and observational
studies. The teacher can model examples of each type and discuss their key points while the students take
two-column notes. It’s important for the teacher to emphasize the role of randomization in studies. A
summary of these methods is located at the following website.

SEE: Making Inferences and Justifying Conclusions HSS-IC.B.3

Practice and Application (time: 15 minutes)
In pairs or groups, students will create their own examples of what a sample survey, experiment, and
observational study might look like. Further differentiation could ask the students to make studies that
contain mistakes, and swap their ideas with classmates to have the other students find the mistakes and
make suggestions.

Review and Assessment (time: 15 minutes)
Students should share their examples and as a class create an exemplar for a sample survey, experiment,
and observational study and record it on a poster.

Lesson 4
The Chi-Squared Test (2 Days)

Goal
Students will compute chi-squared statistics and identify whether the null hypothesis should be rejected
or not. They will compare what actually happened to what a model predicts will happen (S-IC.2).

Lesson 4—DAY 1:
Do Now (time: 5 minutes)
Students will write in response to this prompt:

Suppose you open a large bag of M&M candies and immediately pick out all the orange ones—your
favorite color—and put them in a jar. Your little sister gets mad and says, “No fair! You’ve got 25% of
all the M&Ms!” You count 110 orange M&Ms in your jar. Your sister says there were 500 M&Ms to
begin with. Did you really take 25%? More than 25%? Less than 25%? How can you tell?
Hook (time: 10 minutes)
The teacher will explain:

According to some internet sources, M&M candies are manufactured with this color distribution:
24% blue, 13% brown, 16% green, 20% orange, 13% red, and 14% yellow.

Note: Many years ago, the Mars Company published the color distribution on their website. They no longer do this and there is no way to verify the current color distribution. The given color distribution is based on a communication with the Mars Company from 2008. It is possible that they have changed their color distribution since then.

The teacher will ask:

How can we tell if these numbers are valid? Students will discuss what it might look like to have this distribution of M&M colors. For example:

Are there always going to be the same number in every bag? Does this mean we will definitely have 24% blue M&Ms in any bag we open? What if we expected 14% yellow and got 13% instead? How far away from the expected value does it have to be before we decide our bag is unusual or the M&M production numbers are not accurate?

A “Breakdown Chart of Color Frequency for M&Ms” Activity Sheet is located on p. 6.14.6 in the Supplement.

In the discussion, students should be able to identify possible sources of variation in the M&M packaging process. The class can watch the video “Watch how Mars makes M&Ms” to see how M&M candies are manufactured and to try to notice how variation in color distribution would occur. In this discussion with the students, variation due to chance should be the key concept: in any process, we expect to get variation. The more steps there are in the process, the greater the chance of variation. It doesn’t mean that something is wrong with the process. It doesn’t mean that the company lied about the color distribution.

SEE: Watch how Mars makes M&Ms
www.youtube.com/watch?v=xVJTLz7Z3Ds

Presentation (time: 40 minutes)
The teacher will continue by asking:

Then the question becomes, how do we decide when something is too far off to be explained by chance?

The teacher will explain that there is a statistical test that will help us answer this question. It is called the Chi-Squared ($\chi^2$) Test for Goodness of Fit. We find the chi-squared ($\chi^2$) statistic to see how far our data are from what we expect. The larger the value of the statistic, the less well our data fit what we expected. (In this case we predict that in a bag of M&Ms there will be 24% blue, 13% brown, etc.).

The chi-squared statistic looks like this:

$$\chi^2 = \sum \left( \frac{(\text{observed count}-\text{expected count})^2}{\text{expected count}} \right)$$

The teacher will then add chi-squared to the Word Wall and lead students through a chi-squared calculation using the data table in the “Chi-Squared Calculation of Color Frequency for M&Ms” Activity Sheet on p. 6.14.7. of the Supplement. It helps to use a table to keep the data organized. In the example, the chi-squared statistic is the final cell of the table. Have students fill in a blank table as partners or individually
using a calculator. After students fill in the first two columns, ask them to guess if they think the expected percentages match what was observed. It is useful to have students guess first because most of the time they get it wrong, which helps demonstrate why we need objective measures to test our predictions.

This is a good point for the teacher to introduce the idea of a hypothesis in statistics, as follows:

A hypothesis is an attempt to explain something. In the world of statistics, we are always wondering if something can be explained just by chance variation, or if something else is going on to explain the result we observe. So what do you think about the M&M results? They don’t match up exactly with the color distribution we expected. But we’ve also seen how the bags of M&Ms are produced at the factory and we know that there will be some variation. We have two competing explanations here:

- The null hypothesis would say, “Hey your bag is fine. It’s what we would expect due to chance variation.”
- The alternative hypothesis would say, “Either this is a really unusual bag of M&Ms or the percentages they published about color distribution are just wrong. The distribution we got is so far off from what we would expect that we can’t explain it with chance variation.”

So which hypothesis do you think we should go with today? The null hypothesis or the alternative hypothesis?

Students can debate these questions as the teacher adds the terms to the Word Wall. They may bring up the chi-squared statistic and ask how it helps us decide which hypothesis makes more sense:

We’ve computed this number, but what do we do with it?

The teacher will ask:

Is the chi-squared statistic large enough to reject the null hypothesis, large enough to decide that something strange is going on here?

The example given shows a statistic of 6.89. The larger the number, the further the results are from what we would expect. However, how can we tell what is “large”? We need a chi-squared table and to know the number of degrees of freedom in the test. The degrees of freedom are equal to one less than the number of categories. In this case the degrees of freedom are equal to 5 because there are 6 colors. (See the Chi-Squared Table Activity Sheets on p. 6.14.8 of the Supplement.) Using 5 degrees of freedom, students can see that a chi-squared of 6.89 is between p=.20 and p=.25. The p value represents the probability that the result can be explained by chance. With that probability between 20% and 25%, we have to acknowledge that chance could easily be the culprit here. Usually, we only reject the null hypothesis if the probability is less than 5%. For this to happen, we would need a chi-squared value of 11.07 or greater. We don’t have that. Therefore, the null hypothesis is a reasonable explanation of the color distribution we observed in this bag of candies.

Lesson 4—DAY 2:

Presentation (time: 10 minutes)

The class will summarize the information from Day 1. For example, students can present their tables and explain what the data represent. Students should be able to articulate the idea that they are trying to decide if their result could be explained by variation due to chance. The teacher should then introduce the vocabulary of proportionality (disproportionate, overrepresented, underrepresented) in the context of populations and samples.
**Practice and Application** (time: 25 minutes)
Students will replicate the M&Ms experiment with several snack-size M&M bags and compose a brief report about the activity, incorporating the vocabulary of proportionality. For example, students may observe that one of their bags was “disproportionately red” or that “brown M&Ms were overrepresented” in a particular bag. Students can then create a report using different media such as a written summary, computer-generated graphs and charts, a poster, and so on.

**Review and Assessment** (time: 20 minutes)
Working in pairs, students will compare the results of the large bag M&M experiment (Day 1) with the results of the snack bag experiment (Day 2). Some questions they should consider:

- Why does sample size matter?
- Why are we more likely to see disproportionality in the smaller bags?
- How does this affect our ability to draw inferences?

Students should explain the results of the comparison in the form of poster presentations, an ELMO, or oral sharing of results. Students will then use this information to respond to this question:

- Why it is important to interpret data carefully?

**Extension**
Students will create their own data and find the chi-squared statistic associated with it. They could have a partner (or the class) try to guess, based on just the observed and expected values, whether the null hypothesis should be rejected, that is, whether it reaches the .05 significance level. How far from the expected value can you get before the null hypothesis is rejected?

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**Lesson 5**

Sample Size, Margin of Error, and Confidence Level (2 Days)

**Goal**
Students will explain how sampling can be used to estimate a value in a population. Students will articulate the relationship among sample size, margin of error, and confidence level. (S-IC.4)

**Lesson 5—DAY 1**

**Do Now** (time: 10 minutes)
Students will compute the probability of flipping a coin six times and getting heads each time. If time allows, they should next compute the probability of flipping a coin six times and getting heads five out of the six times.

**Note:** This statistics unit is intended to follow the unit on probability, so this exercise should be review.

**Hook** (time: 5 minutes)
The teacher will bring up some issues that might be the subject of an opinion poll, such as preferences in an election or feelings on gun control. The teacher should elicit some suggestions of other issues from students. Then the teacher should ask students if they believe that a random sample of 2,000 people could
be used to get a fairly accurate picture of how the entire country feels about the issue. Most students (indeed most adults) will think that 2,000 people is too small of a sample. A widespread misconception about sampling is that the sample should be some fraction of the population, and 2,000 seems like a very small fraction of 300 million. In fact, the sample size is not determined that way. It’s more related to probability. That will be investigated in this lesson.

**Presentation** (time: 25 minutes)
The teacher will present students with a list of the different probabilities for flipping a coin six times:

- 0 heads: $\frac{1}{64}$
- 1 head: $\frac{6}{64}$
- 2 heads: $\frac{15}{64}$
- 3 heads: $\frac{20}{64}$
- 4 heads: $\frac{15}{64}$
- 5 heads: $\frac{6}{64}$
- 6 heads: $\frac{1}{64}$

The teacher should have a student come to the board and make a quick sketch of a bar graph to show the probabilities (number of heads is the quantitative variable on the x-axis). The shape of the bar graph will roughly approximate a normal distribution. Students should be able to articulate why the probability of 2 heads is greater than the probability of 1 head: because there are more ways for that result to happen.

**Note:** Do not take time at this point to develop the combinatorics of counting the number of ways. That is a lesson in itself and belongs in the unit on probability.

The main idea here is for students to be reminded of where the normal distribution comes from, and to realize that while the mean value is the most likely outcome, other results are certainly possible. Indeed, we should expect to see those less likely results from time to time. So, it’s not surprising to flip a coin and get a streak of heads or tails. The teacher should show students the history of coin flips from the Super Bowl (see chart below) and have them look for streaks.

The teacher will explain:

In statistics, it’s important to keep in mind the shape of the normal distribution. Most of the distribution is in the middle, but there is definitely some of it in the tails too. This is important when we are designing opinion polls. We know that even if we choose people randomly to respond to our poll, there is some probability that we will happen to pick a distorted sample.

### The First Fifty Years of Super Bowl Coin Tosses

<table>
<thead>
<tr>
<th></th>
<th>Heads</th>
<th>XI</th>
<th>Tails</th>
<th>XX</th>
<th>Tails</th>
<th>XXX</th>
<th>Heads</th>
<th>XLI</th>
<th>Heads</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Tails</td>
<td>XII</td>
<td>Heads</td>
<td>XXII</td>
<td>Heads</td>
<td>XXXII</td>
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<td>III</td>
<td>Tails</td>
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<td>XXXIV</td>
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<td>XLIV</td>
<td>Heads</td>
</tr>
<tr>
<td>IV</td>
<td>Tails</td>
<td>XV</td>
<td>Tails</td>
<td>XXV</td>
<td>Heads</td>
<td>XXXV</td>
<td>Tails</td>
<td>XLV</td>
<td>Heads</td>
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<td>V</td>
<td>Heads</td>
<td>XVI</td>
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<td>VI</td>
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<td>XVII</td>
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<td>XXXVII</td>
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<td>VII</td>
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<td>Tails</td>
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</tbody>
</table>

The teacher should set up a simulation using colored plastic cubes in an opaque bag. The bag should contain 60 red cubes and 40 blue cubes, but students should not be aware in advance what the distribution is. Give the colors a context of preference or opinion—such as Coke drinkers vs Pepsi drinkers, Red Sox fans vs. Yankees fans, or whatever might appeal to the students. Then have the students do samples of 10 cubes, picked one at a time. (Since the population represented in the bag is fairly small, it’s important to sample with replacement—put each cube back in the bag after picking.) And no peeking in the bag! Students should repeat the sampling process 8 times, recording their results. The teacher should then ask students to speculate about the composition of the bag:

Do you think it’s 50-50 red/blue? Or maybe more red? Or more blue?

How do your sampling results help you decide?

Note: This is the purpose of sampling—to use a sample to estimate the value in the population.

On a piece of 1-inch graph chart paper, the teacher should set up an x-axis to represent the percentage of red (columns marked as 0%, 10%, 20%, 30%, etc..) Students should then come up to the board and use X marks to make a frequency bar graph of the results from their samples. For example, if a student had three samples that were 50% red, he or she should make three X marks (one above the other) in the 50% column.

At this time, the teacher can tell the students that the true distribution is 60% red, 40% blue. Overall, students will likely see more red than blue in their samples, but they will also have some particular samples where that is not the case. In this situation there is approximately 25% probability that a sample of 10 cubes will represent the 60-40 distribution in the bag. But there is also an 11% chance of seeing the opposite—4 red, 6 blue—which would give a very distorted picture of the population. This possibility of a distorted sample raises the issue of sampling error—the difference between the value estimated by the sample and the true value in the population that is being sampled. If a sample showed 40% red, when the bag is really 60% red, that’s a big sampling error.

The table below shows the theoretical probabilities of each possible outcome from a sample in this experiment. The teacher should make a bar graph of these probabilities to present to the students. If there were many students doing the cube experiment, the class’s frequency bar graph would eventually start to resemble the theoretical probability graph.

<table>
<thead>
<tr>
<th>Number of red cubes in sample of 10</th>
<th>Percent red</th>
<th>Probability of that result</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0%</td>
<td>0.000105</td>
</tr>
<tr>
<td>1</td>
<td>10.0%</td>
<td>0.001573</td>
</tr>
<tr>
<td>2</td>
<td>20.0%</td>
<td>0.010617</td>
</tr>
<tr>
<td>3</td>
<td>30.0%</td>
<td>0.042467</td>
</tr>
<tr>
<td>4</td>
<td>40.0%</td>
<td>0.111477</td>
</tr>
<tr>
<td>5</td>
<td>50.0%</td>
<td>0.200658</td>
</tr>
<tr>
<td>6</td>
<td>60.0%</td>
<td>0.250823</td>
</tr>
<tr>
<td>7</td>
<td>70.0%</td>
<td>0.214991</td>
</tr>
<tr>
<td>8</td>
<td>80.0%</td>
<td>0.120932</td>
</tr>
<tr>
<td>9</td>
<td>90.0%</td>
<td>0.040311</td>
</tr>
<tr>
<td>10</td>
<td>100.0%</td>
<td>0.006047</td>
</tr>
</tbody>
</table>
This illustration shows the first 39 results of a simulation with sample size 10, repeated 100 times.

The teacher should ask:

Did anybody get a sample of 10 red cubes? No? Okay, we probably wouldn’t expect to see that today. But it wouldn’t be surprising that someone got a sample of 8 red cubes.

It’s important to realize that sampling error arises even when the method of picking the sample is completely random. It’s all about probability.
The teacher can set up and project this simulation using the “NCTM General Purpose Tools” found at the National Council of Teachers of Mathematics (NCTM) website. It will look similar to the data table pictured above.

**SEE:** NCTM General Purpose Tools

www.nctm.org/Classroom-Resources/Core-Math-Tools/General-Purpose-Tools

The simulation needs to be downloaded. From the menu bar, choose:

*Build ➔ Distribution Events ➔ Random Binomial*

Double click where it says Binomial n = 100 and then type into the text boxes to get a sample size of 10 (called Number of Trials) and a red-cube probability of .6 (called Probability of Success). Use the “Conduct” button to run the simulation, with a large number of samplings. This replicates the experiment the students did, only with more repetitions. This increases the chance of seeing an unusual event, such as a sample where all 10 cubes are red. Scroll up and down in the results table to look for that.

To see a histogram of the simulation results (similar to Simulation Model 1 pictured below) choose:

*Edit ➔ Analyze Results*

A new Data Analysis window will open. Click on the histogram icon, and choose the second option for columns.

**Simulation Model 1**

![Simulation Model 1](image)

**Histogram Settings**

- Min x = 0.0
- Bin Width = 1.0
- Binomial n = 10 & p = 0.6

The teacher will explain:

This histogram shows how probability governs sampling. At 24.7%, the most likely outcome is 6 red cubes, which correctly represents the distribution in the bag. But we also see that sampling error will occur with lots of results at 5 red cubes or 7 red cubes, a fair amount at 4 or 8, and some outside that range too.

So, given that we know we will get sampling error, we have to decide how much error we can tolerate. We might say that a sample with 80% red is just too far off for our purposes. Maybe we will decide that we need to be within 10% of the true value. This is the margin of error—a plus or minus value.
of acceptable sampling error. (The term should be added to the Word Wall.) It’s like setting up two vertical fences on the distribution and then saying we want to be inside this region. (The teacher should draw fences on the histogram to show the region between 5 and 7 red cubes.)

Now, do we think that we will be inside that region when we draw our sample out of the bag? What are the chances that our sample will be somewhere in the region between 50% red and 70% red? Checking the shaded region of the probability table above we see there is about a 67% chance of getting a result in the desired range. Are we happy with that? Are we confident that our sample of ten cubes gives a good representation of the population in the bag?

The class can now discuss the idea of confidence level, the probability that a sample is within a certain margin of error. (Like all other new terms, this one should be added to the Word Wall.) If we are using a margin of error of plus or minus 10% for our cube samples, then we would say the confidence level is 67%, which is about two-thirds. Are we happy with an opinion survey that is right two-thirds of the time? Or do we want to be more confident?

Students might suggest that it would be better to be 90% confident. That would mean the poll is likely to be distorted 1 out of 10 times—is that acceptable? If you owned a beverage company, would you pay for that survey? Most often today, opinion polls have a 95% confidence level. That sounds great—95%—but even that will be distorted 1 in 20 times. Still, it’s certainly better than 67% confidence.

What would we have to do in order to get a higher confidence level for our cube samples? (Students may suggest that we could widen the margin of error.) If we increase it to plus or minus 30%, then we would have 98% of the probability captured in our desired region. That’s a high confidence level, but it’s a terrible margin of error! It’s saying that any result between 30% red and 90% red is okay with us. That’s a pretty useless survey. So what do we do in order to maintain our confidence level while also improving our margin of error?

Students will probably suggest that we should increase the sample size. This, too, can be viewed through the lens of probability. If you only choose 10 cubes, the chance of getting something very distorted is considerable. As you sample more cubes, the probability of distortion goes down. (For example, with a

Simulation Model 2

Histogram Settings
Min x = 10.0
Bin Width = 2.0
Binomial n = 50 & p = 0.6

NCTM General Purpose Tools.
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sample of 10 cubes the chances of getting all red is \(0.6^{10}\), but with a sample of 50 cubes, it’s \(0.6^{50}\)—much, much smaller.)

A new histogram (Simulation Model 2) is shown on p. 6.12.23, where sample size is 50 instead of 10:

- With the larger sample size there are now more results between 50% red and 70% red (from 25 red cubes to 35 red cubes).
- So, increasing the sample size does increase our confidence.

**Practice and Application** (time: 15 minutes)

Students should go online to find an opinion poll of interest. They should check to see if the poll reports its sample size, margin of error, and confidence level. Frequently, polls do not report confidence level, in which case students can assume it is 95%.

The students should be reminded that samples are just estimates of what study designers are really interested in: the population. When students find a poll of interest, they should practice using the plus or minus idea of the margin of error, so that they can find the range from low estimate to high estimate. They might look at Rasmussen’s “Right Direction or Wrong Track” poll which has been conducted weekly for years:

- **SEE:** Rasmussen Reports—Right Direction or Wrong Track Poll
  - www.rasmussenreports.com/public_content/politics/mood_of_america/right_direction_or_wrong_track

The teacher should explain:

- The poll has a margin of error of 2 percentage points with a 95% confidence level. So, if the poll finds this week that 29% of people in the sample believe that the country is going in the right direction, there is a 95% probability that the true value in the American population is between 27% and 31%.
- Now suppose next week Rasmussen’s survey shows that 30% of people in the sample believe the country is going in the right direction. What can we conclude? Did it go up? Or are we just seeing sampling variation? Because Rasmussen shows the data for multiple weeks, students can look to see if they can spot any trends and perhaps tie those trends to events occurring at the time.
- When looking for a trend, students need to ask themselves: Is this a real change, or just variation due to chance? That is an essential question in statistics.

Students should look for examples of polls that are or were “too close to call”—that is, the results are within the margin of error. This happened in June 2016 with the British referendum on leaving the European Union. Before the vote, opinion polls showed a slight preference for remaining within the EU, but the poll results were within the margin of error. As it turned out, the exit vote (Brexit) won the election.

As students look for various polls, they should take notes and bookmark URLs that they may want to return to. This online research will be continued and presented on the following day of the lesson.

**Lesson 5—DAY 2:**

**Do Now** (time: 5 minutes)

The teacher should project the poll results chart from the *Real Clear Politics* website.

- **SEE:** Election Other—Presidential Trump Job Approval
  - www.realclearpolitics.com/epolls/other/president_trump_job_approval-6179.html
This page shows the results of multiple polls from March 2017 (use Google to find the URL for a more recent version of the poll). If there is a current election, it might be better to use polls showing the support for each candidate in the race. The only requirement is to find a set of different polls that are all about the same thing.

Students should write individually in response to this prompt:

If these polls were all conducted in the same time period, how is it possible that they show different results? What could explain this?

**Hook** (time: 10 minutes)

The class should discuss their written responses about why the results of the polls differ from each other. Students may raise issues about the quality of the polling—maybe some polling organizations are better than others at getting a true random sample. This would be a good time for the teacher to talk about some of the factors that can affect randomness: For a sample to be truly random, it must be true that each person in the population has an equal probability of being selected for the poll. The class can brainstorm different ways that the sample could be biased. For example:

- Does the polling organization use land lines exclusively?
- Which groups of people might be less likely to have a land line?
- Are some groups more likely to answer the phone and to be willing to participate in the survey?

Bias may also be introduced by the way questions are asked or how the pollster interacts with the respondent. Students may question if the different polls were conducted in different regions of the country or if they are all nationwide polls. Would that make a difference in the results?

Students might wonder about the sample sizes showing A, RV, and LV. These abbreviations stand for Adults, Registered Voters and Likely Voters.

- Does it matter which group you ask?
- Could that explain the differences in the result of the polls?

**Note:** For teacher background, one polling company explains its sampling methodology here:

**SEE:** Rasmussen Reports—Methodology

www.rasmussenreports.com/public_content/about_us/methodology

Also, students may want to discuss how the polls appeared to be wrong in the 2016 presidential election. A good resource for understanding this can be found at Nate Silver’s fivethirtyeight.com:

**SEE:** Can You Trust Trump’s Approval Ratings?


**Presentation** (time: 10 minutes)

Ultimately, though, it’s important that students realize the importance of margin of error in this context. Even if the samples were completely random, we should expect to see variation in the samples just due to chance.

For most of the polls, the margin of error is 3%. With the Real Clear Politics chart projected on the whiteboard, different students should come up and write next to the poll the range of the estimate. For example, if a poll shows Trump’s approval at 42%, the 3% margin of error means the estimate has a range from 39% to 45%. Once all of the ranges have been posted, the class should discuss if the variation in results can be explained just due to margin of error.
Students also need to remember confidence level. We are only 95% confident that the population value we are trying to estimate falls within the margin of error.

The teacher should then ask students to look at the number of people who were interviewed for each poll. Students will be skeptical of the sample sizes here. There are about 150 million registered voters in the U.S. How can a sample of 1,000 possibly represent a population of 150 million? This will raise the misconception that a sample needs to be some fraction of a population. That’s not true. The teacher should refer students back to the bag of red and blue cubes. The chance of picking a red cube was 60%. If the bag contained 100 million cubes (60 million red, 40 million blue), the chance of picking a red cube would still be 60%. So it doesn’t matter how many cubes are in the bag. It’s the probability that matters.

Note: Population size matters only if individuals are being picked without replacement. For small populations, removal of an individual will affect the mixture remaining in the bag and change the probability.

**Practice and Application** (time: 15 minutes)

Students should go online and experiment with the sample size calculator found below.

SEE: Sample Size Calculator
www.raosoft.com/samplesize.html

They should start by putting in the margin of error and confidence level information from the Rasmussen “Right Direction or Wrong Track” poll and see if the calculator outputs Rasmussen’s sample size of 2,500 people. Students can then try changing the margin of error or the confidence level and see how the sample size changes in response. After experimenting with the sample size calculator, students should return to the online research from the previous day.

SEE: Rasmussen Reports—Right Direction or Wrong Track Poll
www.rasmussenreports.com/public_content/politics/mood_of_america/right_direction_or_wrong_track

**Review and Assessment** (time: 15 minutes)

Students should present information about their polls of interest. Students should project their polls of interest for the class (or summarize them, if no projector is available) and explain what can be inferred from the poll results, given the margin of error and the confidence level. Is the poll too close to call? If you were the person paying for the poll, would you feel satisfied with the results or would you want to pay for a more accurate poll?

**Extension**

Students can experiment with the formula for margin of error:

\[
\text{margin of error} = z \sqrt{\frac{p \cdot (1 - p)}{\text{sample size}}}
\]

where \( p \) represents the proportion in the general population. With the cube experiment, \( p \) would be .6 because the bag contained 60% red cubes.

The \( z \) value represents a number of standard deviations related to the confidence level. For a 95% confidence level, the \( z \) is 1.96. This comes from the normal distribution curve, where 1σ is 1 standard deviation (see the Standard Deviation Diagram illustration on the next page, p. 6.12.27).
With a diagram like this, students can add up the percentages going from $-2\sigma$ to $2\sigma$. They will get a sum of 95.4% and see that using 2 standard deviations below and above the mean gives you a little more than 95% of the area under the curve. Thus, it’s believable that going from $-1.96\sigma$ to $1.96\sigma$ gives you closer to 95% exactly. So that is where the 1.96 number comes from. For a 99% confidence level, the $z$ value is 2.58. Other $z$ values can be found at this website:

SEE: Checking Out Statistical Confidence Interval Critical Values
www.dummies.com/how-to/content/checking-out-statistical-confidence-interval-criti.html

Students should try plugging values into the formula for margin of error and then see if the results match the values output by the sample size calculator:

SEE: Sample Size Calculator
www.raosoft.com/samplesize.html

It is not worthwhile to derive all aspects of this formula with students, but they could take some time to understand the implications of the formula and see that the relationships make sense. In the formula, it’s important to notice the inverse relationship between Margin of Error and Sample Size. Holding the value of $p$ constant (at say, .6), students should do a computation to see what happens to the value of the margin of error as sample size changes. If you double the size of your sample, what happens to the margin of error? What if you quadruple the size of your sample? (Margin of error will be cut in half.) Students should be able to explain in the context of sampling why a bigger sample would reduce the margin of error.
Lesson 6

Experimental Studies (2 Days)

Goal
Students will use data and simulation to compare two treatments and determine whether the differences are significant. (S-IC.5)

Lesson 6—DAY 1:

Do Now (time: 5 minutes)
The teacher will present students with a chi-squared table and the following information and ask students if the data are significant at the .05 level. If they are not, students will determine what the statistic would need to be to make it significant.

1. With 10 degrees of freedom and \( c^2 = 17.5 \) (No; \( \chi^2 \) must be \( \geq 18.31 \))
2. With 3 degrees of freedom and \( c^2 = 6.5 \) (Yes)

Hook (time: 5 minutes)
The teacher will ask students to think about two different brands of some product, such as athletic shoes. Is there really a difference in the brands? How might we use data in order to decide this?

Students will need to arrive at some kind of quantitative variable that can be used to measure the difference. They will also need to think about going beyond anecdotal evidence. For example, how could athletic shoes' comfort or durability be measured? This discussion will lead into the example provided in the Presentation.

Presentation (time: 20 minutes)

SEE: NCTM General Purpose Tools (Needs to be downloaded—see Lesson 5)
www.nctm.org/Classroom-Resources/Core-Math-Tools/General-Purpose-Tools/

The teacher may use pre-loaded data from the NCTM Core Math Data Analysis Tool (Data → Univariate Data → Quantitative Data → A – K → Battery Life) to project a data set comparing two groups of batteries (see chart on p. 6.12.29).

The teacher should first point out to students the variability within each group. The class can discuss why different batteries of the same brand would fail at different times. Then students should speculate on which brand is better, based on the data shown. Some students may focus in on extreme cases, like the AlwaysReady battery that failed at 19 hours or the one that lasted 203 hours. This would be a good time for the teacher to bring up the idea of anecdotal evidence. A customer who bought the 19-hour battery is going to tell a very different story from the customer who bought the 203-hour battery. But both of those stories can be explained in the context of variability.

Given the variability in the data, students may decide that it’s hard to tell which brand is better, if there is really any difference at all. Students may have a belief that there are always significant differences between brands, so it’s important for the teacher to bring up the possibility that two groups may not have a significant difference. The class should arrive at the idea that taking the mean of each group would help us
compare, so we can decide if there is indeed a significant difference between the two brands of batteries, based on the sample data that we have.

To see the mean battery life for each group, go to Statistics ➔ Descriptive. This will show that the AlwaysReady batteries in the sample averaged 98.02 hours, while the ToughCell batteries averaged 105.22 hours. In the descriptive statistics, students may notice that the AlwaysReady batteries have much more variability, as measured by standard deviation.

So, there is a difference in the means: 98.02 – 105.22, or -7.2 hours. (The NCTM tool will always subtract Column A’s mean from Column B’s mean, so it may come out negative.) The teacher should ask:

Now the question is: Is -7.2 hours a significant difference? We looked at 40 batteries of each type. Maybe just by chance we happened to get a lower mean in the AlwaysReady group. What if we just mixed the two groups together and then picked out 40 batteries at random? Is it possible we would end up with a similar result—where the two groups differ by 7.2 hours or more?

This is the essential question in statistics: Could this result have occurred due to chance?

The null hypothesis would say that there is no real difference between AlwaysReady and ToughCell; any apparent difference in the data is due to chance. We need to decide if we can reject the null hypothesis or not. The NCTM tool will do this experiment for us. It will take the two groups of batteries, mix them up, and pick out two samples of 40 batteries at random.

To demonstrate this, go to Statistics ➔ Randomization Distribution. Choose OK for Column 1 and Column 2. A new window will open, showing that the difference in the means is -7.2. Note that the numbers in the AlwaysReady column are red and the numbers in the ToughCell column are blue. Now hit the green triangle button (“Start”). The first column will fill slowly, showing

<table>
<thead>
<tr>
<th>BATTERY LIFE (Life in hours of two brands of batteries)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AlwaysReady Battery</td>
</tr>
<tr>
<td>1</td>
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<td>2</td>
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<tr>
<td>3</td>
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<td>25</td>
</tr>
</tbody>
</table>

Source: Data analysis of CD Rom Bright G (2003) Reston VA. NCTM.
that there is now a mixture of red and blue numbers in the first group. These have been chosen randomly. The remaining 40 batteries will be put into the second column. The means will be computed and the difference in the means will be displayed at the bottom of the window. Is the difference -7.2? Probably not. Hit the green button again and a new experiment will be conducted. If you do this ten times, you have a pretty good chance of seeing a difference that is -7.2 or less. When that happens, it’s important for students to realize that it occurred just due to chance. It was just that particular sample of 40 batteries that happened to come out that way. It’s not the brand making the difference--the brands were mixed together for this experiment.

Students may point out that, okay, it happened that one time, but it doesn’t seem to happen very often. So the next question is: How often would something like that happen if we just choose 40 batteries at random? To decide that, we need a simulation with multiple trials. To do this, you can hit the double blue triangle “Faster” button (you can hit it more than once to speed it up).

Histogram 1 below shows the result of doing the same random sampling procedure over 20,000 times. (To see the frequencies in percentages, go to Options and put a checkmark in Relative Frequencies and Label Bars.)

The x-axis of this histogram is the difference in the means. The vertical red line is positioned at -7.2, which was the difference we saw originally before the two groups were shuffled together. Students should now use the histogram to help decide this question:

How unusual is it to see a difference of -7.2 or less? Could it happen by chance?

It’s certainly not the most likely event, but it’s also not that extreme. Could we be 95% confident that it won’t happen? No. If we add up the percentages to the left of the red line, we get somewhere around 11%. There’s about a 1 in 9 chance that we will see this result not due to a difference in battery brand, but just due to which 40 batteries we happen to pick. So we cannot reject the null hypothesis and the difference of -7.2 hours would not be considered significant at a 95% confidence level.
The teacher should now introduce the helicopter experiment that will start today and finish tomorrow:

We are always asking: Is there a difference? Sometimes there is and sometimes there isn’t. Let’s try making a product of our own—paper helicopters. We will make two different models, one with a longer wing length, and then see if wing length makes a difference in the flight time of the helicopter. Show students an example of each model.

We will need to make quite a few of the helicopters, of course, because we don’t want to rely on anecdotal evidence.

What’s your intuition about the wing length? Will it make a difference? If so, which model will fly for a longer time? What makes you think that?

**Practice and Application** (time: 25 minutes)

The teacher will give students the “Drawing a Conclusion from an Experiment” Activity Sheet for Lesson 6 found on pp. 6.14.9-6.4.14 of the Supplement. For the remainder of the class session, students should build the helicopters as directed in the handout and then gather their flight time data. It’s important that the teacher looks at the teacher guide for the helicopter experiment, found at the EngageNY Algebra II website. This activity has been modified to use the NCTM General Purpose Tools. The Teacher’s Guide can be accessed at the second URL listed below.

**SEE:** Algebra II, Module 4 (EngageNY)
www.engageny.org/resource/algebra-ii-module-4-topic-d-lesson-28
Teacher’s Guide (EngageNY)

**Lesson 6—DAY 2:**

**Do Now** (time: 10 minutes)

The teacher will explain that a study compared two drugs in the treatment of migraines. The quantitative variable was the number of minutes it took for patients to feel relief from pain. Researchers found that there was a difference in the means: On average, Drug A took 16.85 minutes longer to provide relief than Drug B did. Is this a significant difference? Or might it just be due to chance? Students should look at the results of the simulation illustrated in Histogram 2 (see p. 6.12.32) and write in response to the following prompt:

Based on the results of this simulation, do you believe you can reject the null hypothesis?
Can we state with 95% confidence that there is a true difference between Drug A and Drug B?

**Presentation** (time: 10 minutes)

The class will discuss what they wrote in the Do Now exercise. Students need to realize that this distribution suggests that we can reject the null hypothesis. There does appear to be a significant difference between Drug A and Drug B. It’s unlikely that the 16.85 difference in the means occurred due to chance. We see from the histogram that such a great difference occurred only 1.2% of the time (the area to the right of the red line). Students may ask how you know when you should go to the right of the red line and when you should go to the left. It depends which tail you are on. They are always looking for the more extreme cases, so they should go in the direction that is farther from the mean.
Practice and Application (time: 25 minutes)

Students will continue with the previous day’s helicopter experiment. If they did not finish gathering data on the previous day, they should clearly mark which data came from which day. It’s possible that some conditions will have changed (such as humidity or temperature). Today, they will focus on analyzing their data using the NCTM General Purpose Tools to do a randomization distribution. They should follow the instructions for Part 3 and Part 4, where they analyze their data and state their final conclusions.

Review and Assessment (time: 10 minutes)

Students will complete an Exit Ticket by writing in response to this prompt:

Explain how constructing a randomization distribution helped you decide if wing length has an effect on flight time.

Students could write, draw, or use technology to summarize and present their findings to the class.

Lesson 7

Evaluating Statistical Studies

Goal
Students will review and evaluate reports based on data. (S-IC.6)

Do Now (time: 5 minutes)

Students will write and/or diagram responses to the following problem:

Suppose you see a research study showing that the average rate for typing text messages is 10 words per minute. Does that result seem believable to you? List three reasons why or why not.
Hook (time: 15 minutes)
Students should share their responses from the Do Now, and the teacher should record their answers on the board or chart paper. Responses may well bring up anecdotal evidence (“I can text way faster than that!”), so this is a good time to talk about the need for sampling to go beyond anecdotal evidence. Students may refer to groups of people such as their family or friends, or they may say something like “That speed is slow because they just asked old people!” This provides a good opportunity to talk about the importance of random sampling. Was it really true that each person in the population had an equal chance of being selected for the study? During the discussion, the teacher should sketch a distribution curve on the board to show the idea of variability. We use sampling because we want to capture that variability.

If time allows, students can debate whether they think a bell-shaped curve is appropriate or not when texting speed is the quantitative variable. For this situation, the bell curve is probably not right. The true distribution is likely to have a long right tail, as there will be some lightning fast texters whose speed is very far from the mean.

Presentation (time: 15 minutes)
The teacher will explain that this introduction to statistics should allow students to critique examples of studies or reports they see in the news. The class should review the Sample Assignments on the Shmoop website.

See: Making Inferences and Justifying Conclusions HSS-IC.B.6

Note: The first two questions on Shmoop refer to the concept of standard deviation, which students have not seen since Algebra 1. The teacher should stick to a fairly informal concept of standard deviation as a measure of spread. If a data set has a large standard deviation, then the distribution curve includes lots of data far from the mean. If a data set has a small standard deviation, the data will be clustered very close to the mean.

Practice and Application (time: 20 minutes)
Students will go to news websites (such as the CBS News website below) and find examples of studies or return to the General Social Survey website.

See: CBS News Polls
www.cbsnews.com/feature/cbs-news-polls
General Social Survey
http://gss.norc.org/

They should find either one study to critique in depth or several studies and explain something they thought was important and/or something they would change. Students should use the following checklist to analyze the studies:

What type of sampling method was used?
Who was sampled?
What population did the sample represent?
What statistics were presented?
Review and Assessment (time: 10 minutes)
Students will report their findings to the class. Students should revise their analyses based on suggestions from the class.

CULMINATING LESSONS
Includes the Performance Task, i.e., Summative Assessment—measuring the achievement of learning objectives

Lesson 8
Reviewing the Types of Statistical Studies

Goal
Students will connect research scenarios to statistical study designs (sample survey, observational, experimental) and types of variables (categorical and quantitative).

Do Now (time: 5 minutes)
Students will sort the following list of parameters into categorical and quantitative lists: zip code, age, income, gender, heart rate, marital status. They should justify their answers.

Hook (time: 5 minutes)
The teacher will remind students that they will be designing statistical studies for the unit’s Performance Task and ask them to discuss the ideas they have been considering for their projects, including the questions they would like to find answers to. The teacher should remind them that designing a study includes selecting a method and identifying variables. The purpose of this lesson is to review the types.

Presentation (time: 15 minutes)
The teacher will show the Khan Academy video on statistical study types. The video includes subtitles, and a transcript is available if needed. The teacher should pause the video after each type of study (sample, observational, experimental) is explained to discuss its essential features and purpose. Creating a class anchor chart based on these discussions would be helpful for the next activity.

\[
\text{SEE: Khan Academy—Types of statistical studies (video)}
\]
\[
\text{www.khanacademy.org/math/probability/statistical-studies/types-of-studies/v/types-statistical-studies}
\]

Practice and Application (time: 20 minutes)
Working in pairs, students will review and discuss the research scenarios on the Khan Academy “Types of statistical studies” page and respond to the questions provided.

\[
\text{SEE: Khan Academy—Types of statistical studies (research scenarios)}
\]
\[
\text{www.khanacademy.org/math/probability/statistical-studies/types-of-studies/e/types-of-statistical-studies}
\]

The questions focus not only on the types of studies conducted but also on whether the designs were appropriate and what conclusions can be drawn from them. Students should discuss and answer the
Tracking Table Example

<table>
<thead>
<tr>
<th>Study Subject</th>
<th>Study Type</th>
<th>Variables (Types)</th>
<th>Null Hypothesis</th>
<th>Alternative Hypothesis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effect of eating a banana before running a marathon on performance in the marathon</td>
<td>Experimental</td>
<td>Banana/No Banana (categorical) Running Time (quantitative)</td>
<td>Eating a banana before running a marathon has no impact on performance in the marathon</td>
<td>Eating a banana before running a marathon influences performance in the marathon</td>
</tr>
</tbody>
</table>

Source: Khan Academy. www.khanacademy.org

Questions. If they get them wrong, they can try again. The site also provides hints to help build understanding. The teacher should monitor students’ work and provide support. As students work through the scenarios, they should keep track of the studies using a table like the tracking table example (above) from the Khan Academy website.

**Review and Assessment** (time: 10 minutes)
The teacher will conduct a brief review of the previous activity and invite students to ask questions to clarify their understanding. Then students will individually write and/or draw Exit Tickets in which they explain how each type of study (sample, observational, and experimental) works. They should also identify other factors that are important to designing a quality study (answers should include sample size and randomization).

Lesson 9
Designing and Conducting a Study (2 days)

**Goal**

Students will create a statistical study to test a hypothesis.

**Note:** By this point in the unit students have had a general overview of how statistics can be used to make inferences and justify conclusions. The key take-away message should be that having a question to answer is not enough. Each study needs to be designed to consider factors that might affect the outcome. Students should have been considering topics that interest them throughout the unit.
On Day 1 of this lesson, students should finalize their ideas and discuss them with other students and the teacher to develop specific questions they think could be answered by statistical studies. The types of studies students design will depend in part on how much access they have to subjects and data. On Day 2, they should actually design and conduct a short study to test their questions. This will include making any necessary materials (e.g., a survey or checklist). Examples of experimental studies are included in the lesson.

**Lesson 9—DAY 1:**

**Do Now** (time: 10 minutes)
Students will respond to the following prompt in writing or aloud:

You have heard the expression “practice makes perfect,” but what kind of practice? When you have to learn new information, what kind of practice helps you the most? Students should share and discuss their responses.

**Hook** (time: 10 minutes)
To introduce the process of developing a research question, the teacher will ask students to take the “Africa: Countries Quiz” (use the “strict test” version). Most students will find the quiz very challenging and receive low scores (out of a possible 55 points). The teacher may wish to engage students in a discussion of why Americans know so little about Africa.

SEE: Africa: Countries Quiz

**Presentation** (time: 20 minutes)
The teacher will ask students to consider what kind of practice would help them to improve their performance on this quiz. The website offers two options, which the teacher can demonstrate: In “practice” mode, users just take the quiz again and again, but they get three tries to guess each country and can use the “show me” button if they get stuck. In the more in-depth “study” mode, users see the names of the countries and can click on them individually to learn background information about the population, climate, economy, etc. Which of these methods would lead to greater improvement on the quiz?

The teacher will explain to students that there is a statistical test that can determine if there is a significant difference between these methods: a t-test (a type of test to determine whether there is a difference in the means between two sets of data). The t-test is similar to the chi-squared test in that it helps answer the question, is there a difference between these two groups? We use a t-test instead of the chi-squared test when we are working with a quantitative variable such as a test score.

The teacher should ask students to design a study comparing the two study methods, including the following elements (sample responses are included):

- **Research question:** (Does in-depth study lead to greater gains in country recognition than practice?)
- **Type of study:** (Experimental)
- **Pretest:** (Two randomly assigned groups each take the “strict test” and record scores)
- **Treatment:** (The two groups each practice for 15 minutes using their assigned methods)
- **Post test:** (The two groups take the “strict test” again, record scores, and compute gains)
- **Significance:** (Run a t-test of group gains at http://www.graphpad.com/quickcalcs/ttest1.cfm)
Note: To verify the results of the t-test, the study could also include a simulation (as in Lesson 6) to determine whether the results could be due to chance.

Another example of an experimental study the teacher could share might be a test of physical skill such as tossing markers into a basket from a distance of 10 feet or dropping paper clips into a jar while standing on a chair. In either of these cases, the control group could be individuals that receive no opportunity to practice, while the treatment group could be individuals that do have an opportunity to practice.

The teacher will explain that students will be designing their own statistical studies based on questions they have been considering. Depending on the nature of the questions, the studies could be sample studies, observational studies, or experimental studies.

Note: Experimental studies are probably the most practical in the DYS classroom setting. No matter what the nature of study is, it is important that students consider and prove or disprove the null hypothesis: Could the results be due to chance?

### Graphic organizer format (Practice and Application):

<table>
<thead>
<tr>
<th>Aspect of Study</th>
<th>Detail</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research question</td>
<td>What method of study is more valuable in mastering an identification task, reviewing background information on the subject or practicing the task repeatedly with help?</td>
<td></td>
</tr>
<tr>
<td>Type of study</td>
<td>Experimental</td>
<td>The study will compare two methods of learning by measuring the gains from pre-test to post-test.</td>
</tr>
<tr>
<td>Independent variable</td>
<td>Study method (categorical)</td>
<td>The two categories are reviewing background information and practicing the test repeatedly.</td>
</tr>
<tr>
<td>Dependent variable</td>
<td>Change in score (quantitative)</td>
<td>The outcome for each group will be the mean gain (or loss) in score from pretest to posttest.</td>
</tr>
<tr>
<td>Method</td>
<td>Two randomly assigned groups each take the “strict test” of African country names and record scores. The groups each practice for 15 minutes using their assigned methods. The two groups take the “strict test” again, record scores, and compute gains. The gains are entered under Group 1 and Group 2 in the t-test calculator.</td>
<td></td>
</tr>
<tr>
<td>Statistical test</td>
<td>Standard t-test</td>
<td>Standard t-tests are used to compare means from two groups, such as control and treatment groups.</td>
</tr>
<tr>
<td>Simulation</td>
<td>Randomization distribution</td>
<td>To see if two groups are significantly different, mix them together and then divide them up into two random samples. Check the difference in the means. Do this many times to get a histogram showing what can happen just due to chance. Then look at where your original difference falls on the histogram. Does it seem likely that difference could occur due to chance?</td>
</tr>
</tbody>
</table>
**Practice and Application** (time: 15 minutes)
Students will design their studies using the graphic organizer format shown on p. 6.12.37 (sample responses for the African countries quiz are included).

Students will submit this form as an Exit Ticket at the end of DAY 1, noting any questions or confusion they have about the study. The teacher will review and provide feedback on the student work before the next class.

**Lesson 9—DAY 2**

**Practice and Application** (time: 15-25 minutes)
Students will review the teacher feedback, tweak the design of their studies as needed, and create and gather any needed materials to conduct them. With the teacher’s help, they should create an organizational plan for conducting their studies, serving as each other’s subjects (among others outside the classroom if possible). The plan may involve students’ taking turns collecting data.

**Review and Assessment** (time: 30-40 minutes)
As students administer their studies, they must carefully collect and sort data and use the appropriate statistical techniques to determine whether the results are significant. In the African countries example, this would mean computing the score gain or loss for each member of each group and plugging these values into the t-test calculator as well as running a simulation using the NCTM General Purpose Tools as seen in Lesson 6. At the end of DAY 2, students should submit an Exit Ticket stating and briefly explaining their results.

**Extension**
Depending on the nature of students’ study designs and the resources available to them, the teacher may need to extend the lesson to include more data gathering and analysis time. For example, if students conduct sample surveys, they will need to create, distribute, collect, and analyze paper or online survey forms or interviews. If they conduct observational studies, they will likely need access to the internet to collect data or the ability to interview participants outside the classroom. The teacher will need to make judgments about what kinds of studies are realistic and advise students accordingly.

**Lesson 10**

Presenting and Publishing a Study (2 days)

**Goal**
Students will present their statistical studies and revise their presentations in response to feedback.

**Note:** Day 1 should be used for students to organize their data and present their studies to the class. As students give their presentations, the audience should use a note-taking guide to record important information about each study along with questions they would like to have answered.

On Day 2, students should create posters or written reports finalizing their studies so that they include all the necessary elements to be valid tests of their original questions. These posters and reports can be saved to use as exemplars for the next time the unit is taught.
Lesson 10—DAY 1:

Do Now (time: 5 minutes)
Students will respond to the following prompts in a quick write:

- What did you learn from your study?
- Were your questions answered?
- Were your results significant?

Hook (time: 5 minutes)
The teacher will engage students in a brief discussion of how to report study findings and conclusions in a way that both valid and engaging to an audience. Students will suggest criteria, which may be used later to develop a rubric.

Presentation (time: 15 minutes)
Students will organize their materials and prepare brief oral reports to the class about their research studies. The reports should include research questions, descriptions and rationales for research designs, and analyses of results, including statistical tests or simulations.

Practice and Application (time: 30 minutes)
Students will present their reports to the class, and as they do so, their classmates will take notes that will enhance their own learning as well as providing descriptive feedback to the presenters. For example, students could respond to the following questions about each report:

- What was the presenter trying to learn from the study? What answers did he or she find?
- What kind of study was it? What design features showed that it was an example of this type?
- How did the presenter test the statistical significance of the results? What did he or she conclude?
- What questions do you have about the study? What suggestions for improving the report?

As an Exit Ticket for Day 1, students (and the teacher) will submit their responses to these questions to the presenters as feedback.

Lesson 10—DAY 2:

Review and Assessment (time: 55 minutes)
Using the feedback from the previous day as a guide, students will plan and develop chart-paper posters or written reports detailing their studies, including all of the elements included in the oral reports, but with a more formal presentation, including tables and/or graphs, statistical results, and analysis. Before students begin constructing their posters or reports, the teacher should review all of the requirements (from the bullets in Lesson 9) and criteria for presentation (from this lesson's hook) and may develop a class rubric with students. The teacher should also show some exemplars (work from previous classes or online examples, like those at the website below).

SEE: ASA 2013 Statistical Poster and Project Competition Winners
http://magazine.amstat.org/blog/2013/08/01/poster-and-project
POST–UNIT REFLECTION
On meeting the Learning and Language objectives
Connections to Empower Your Future
UNIT: Data Analysis and Statistics

Future Ready Connections

Teachers are encouraged to use the Future Ready Rubric to evaluate students’ growth in developing communication skills, demonstrating accountability, and the ability to take initiative during activities and projects. Youth have many opportunities to strengthen their communication and listening skills through partner work, group discussions, and the presentation of their Performance Task.

Students will also communicate through writing in their Do Nows and Exit Tickets which can be evaluated for clarity, coherence, and critical thinking. Youth should also be evaluated for initiative and self-direction, especially during activities when they must theorize, conduct research, perform experiments, and draw conclusions from experiments and data sets.

Teachers should reflect on whether or not youth stay on task without prompting and if they push themselves to thoroughly complete each activity, propose theories, answer their own questions, and create a detailed final product instead of only addressing the minimum required information. Teachers should encourage students to reflect on how they demonstrated growth and increased understanding throughout the unit, what they could do to further improve their skills and understanding, and how what they have learned is transferable to other situations and experiences.

Essential Question Connections

The three Essential Questions for this unit ask:

How can I collect, organize, interpret, and display data to investigate a question?

Why is it important to interpret data carefully?

Could the result be due to chance, or is something else going on?

These questions encourage youth to not only interact with data, but reflect on the best practices to collect, analyze, and use data in order to solve a problem or answer a question. Students will need to go beyond simply using data to the more challenging tasks of evaluating and assessing the value of data and then understanding its application to real world settings. By understanding how data can be flawed, misinterpreted, or influenced by the data collection process, students will be more discerning of data and data implications that they come across in their personal, professional, and academic lives.

Teachers can expand on the concept of the validity of data and the need to represent data carefully and honestly by having students research and compare conflicting data reports.

Students can analyze what was different in each report’s processes and conclusions based on the data and then reflect on how or why the report’s writers made these choices. Some topics that students may explore include: benefits of specific supplements, effectiveness of specific diets or exercises, and conflicting claims about climate change causes, or labor market reports.

Career Exploration Connections

STEM-related jobs are a rapidly growing industry in the United States and especially in New England. Teachers can expand on this unit by making connections between the content and career exploration research. Students can research which career fields and industries collect, analyze, and use statistics. Examples include: educational testing and measurements, Food and Drug Administration, finance, risk assessment, marketing.
“The responsibility is placed on each youth to demonstrate their own learning and support the learning of their classmates which allows for interpersonal skill development, leadership skill development, and positive relationships.”

manufacturing, insurance, ecology, and many more. Students can use search engines and keywords or visit the American Statistician Association (ASA) website to identify the career fields.

SEE: American Statistician Association
www.amstat.org/careers/whichindustriesemploystatisticians.cfm

The ASA page will link to different career fields and explain how each career field uses data. Teachers can have students conduct research, create a product (such as a PowerPoint presentation, brochure, poster, etc.) that summarizes the information, and present the information to the class. Students can also use the MassCIS (Massachusetts Career Information) website to research jobs, career fields, industries, and programs of study associated with statistics and data analysis.

SEE: MassCIS
https://portal.masscis.intocareers.org

PYD/CRP Connections

This unit reflects Culturally Responsive Practice by using realistic situations and data sets that are likely familiar to youth (M&Ms, batteries, medications). Students see that math is a universal skill that can be used in the academic, professional, and personal realms. By using realistic situations and easy to understand visuals and comparisons, students will understand that the ability to collect data with correct parameters and a clear understanding of how probability and sample size impacts data is essential. Students can apply their knowledge about data collection, analysis, and reporting to many actions in their lives such as planning a major purchase, deciding on a candidate during an election, or even playing fantasy football.

The application of math skills to meaningful and purposeful contexts allows for authentic outcomes that youth can practice and duplicate independently in the future.

The unit also demonstrates Positive Youth Development by encouraging students to be leaders in the classroom and to be active participants in their learning. Students must conduct experiments and chart results such as during the helicopter experiment. Youth also have the opportunity to work with, support, and learn from each other’s discussions, presentations, and representations of data. The responsibility is placed on each youth to demonstrate their own learning and support the learning of their classmates which allows for interpersonal skill development, leadership skill development, and positive relationships.

For Technical Assistance with Empower Your Future connections and lessons, please request support by submitting a Coaching Request ticket using the Coaching Feature on TeachPoint.
Advanced Statistics Pre-Assessment
Lesson 1

DIRECTIONS: Answer all the following problems. Calculator use IS allowed.

Three hundred twenty-one people were asked how they got to school. Use the table below to find each percent and answer questions 1 to 4 below. Round to the nearest whole percent.

<table>
<thead>
<tr>
<th></th>
<th>Bus</th>
<th>Walk</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>110</td>
<td>85</td>
<td>195</td>
</tr>
<tr>
<td>Female</td>
<td>84</td>
<td>42</td>
<td>126</td>
</tr>
<tr>
<td>Total</td>
<td>194</td>
<td>127</td>
<td>321</td>
</tr>
</tbody>
</table>

1. What percent of the students walk?

2. What percent of the students are male?

3. What percent of the total students are female who ride the bus?

4. What percent of males walk?

5. Find the mean, round to the nearest tenth: 7, 9, 12, 15, 18, 9, 6

For questions 6 and 7, use the following formula to find $\chi$:

$$\chi = \frac{(O - E)^2}{E}$$

6. $O = 10; E = 5$

7. $O = 2; E = 8$
Study the histogram pairs shown in Part A and Part B and answer the questions.

PART A:
The histograms show the distribution of heart rates of randomly selected adult males between the ages of 40 and 45 after 20 minutes of continuous exercise. The adult males were randomly assigned to either a new elliptical machine (Experimental Group) or a traditional treadmill machine (Control Group).

8. Do you think there is a significant difference in average heart rate between the Experimental Group and the Control Group in this trial?
   What did you look at to decide?
PART B:
After participants worked out three times per week for four weeks solely on their assigned machines, participants’ heart rates were collected again after 20 minutes of continuous exercise. The data are shown in the histograms.

![Histogram of Heart Rate after 20 Minutes of Exercise for Experimental Group](image1)

![Histogram of Heart Rate after 20 Minutes of Exercise for Control Group](image2)

9. Do you think there is a significant difference in average heart rate between the Experimental Group and the Control Group in this trial? Justify your answer.
Whether bullying in schools is increasing, as is widely believed, was investigated drawing upon empirical studies undertaken in a wide range of countries in which findings had been published describing its prevalence at different points in time between 1990 and 2009. Results do not support the view that reported bullying in general has increased during this period; in fact, a significant decrease in bullying has been reported in many countries. However, there are some indications that cyber-bullying, as opposed to traditional bullying, has increased, at least during some of this period. The reported decreases in the prevalence of school bullying are consistent with reports of significant but small reductions in peer victimization following the implementation of anti-bullying programs in schools worldwide.

10. What is the topic of this study?

11. During what years did the authors look at data?

12. Does this study support that bullying is increasing?

13. What did the authors do to support their claims?
ANSWER KEY
Lesson 1— Pre-Assessment

1. 40% (127/321)
2. 61% (195/321)
3. 26% (84/321)
4. 44% (85/195)
5. 10.9
6. $\chi = 5$
7. $\chi = 4.5$
8. Students should realize that the two groups in Part A aren’t significantly different. Answers may note that the graphs appear to have a similar mean and spread.
9. The two groups in Part B do appear to be significantly different. Students should note that the experimental group mean is close to 190, while the control group mean is approximately 170.
10. Has bullying changed over the course of 10 years.
11. From 1990-2009
12. No, bullying in general has decreased but cyber bullying has increased.
13. The authors looked at empirical studies conducted in different countries on the prevalence of bullying.
## Breakdown Chart of Color Frequency for M&Ms

### Lesson 4

<table>
<thead>
<tr>
<th>Color</th>
<th>% Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blue</td>
<td>24%</td>
</tr>
<tr>
<td>Brown</td>
<td>13%</td>
</tr>
<tr>
<td>Green</td>
<td>16%</td>
</tr>
<tr>
<td>Orange</td>
<td>20%</td>
</tr>
<tr>
<td>Red</td>
<td>13%</td>
</tr>
<tr>
<td>Yellow</td>
<td>14%</td>
</tr>
<tr>
<td>Total</td>
<td>100%</td>
</tr>
</tbody>
</table>

Color frequency statistics for a 14 oz. package of M&Ms Milk Chocolates. (Mars Company)

### Blank Data Entry Table

<table>
<thead>
<tr>
<th>Color</th>
<th>Observed #</th>
<th>Expected # (Total %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blue</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Brown</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Green</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Orange</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Red</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yellow</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Chi-Squared Calculation of Color Frequency for M&Ms

Lesson 4

Data Table with Values

<table>
<thead>
<tr>
<th>Color</th>
<th>Observed #</th>
<th>Expected # (Total %)</th>
<th>(O – E)</th>
<th>(O – E)^2</th>
<th>(O – E)^2 / E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blue (24%)</td>
<td>9</td>
<td>(.24) (61)=14.64</td>
<td>-5.64</td>
<td>31.81</td>
<td>2.173</td>
</tr>
<tr>
<td>Brown (13%)</td>
<td>7</td>
<td>7.93</td>
<td>-0.93</td>
<td>.86</td>
<td>.108</td>
</tr>
<tr>
<td>Green (16%)</td>
<td>15</td>
<td>9.76</td>
<td>5.24</td>
<td>27.47</td>
<td>2.81</td>
</tr>
<tr>
<td>Orange (20%)</td>
<td>10</td>
<td>12.2</td>
<td>-2.20</td>
<td>4.84</td>
<td>.397</td>
</tr>
<tr>
<td>Red (13%)</td>
<td>8</td>
<td>7.93</td>
<td>.07</td>
<td>.005</td>
<td>.0006</td>
</tr>
<tr>
<td>Yellow (14%)</td>
<td>12</td>
<td>8.54</td>
<td>3.46</td>
<td>11.97</td>
<td>1.402</td>
</tr>
<tr>
<td>Total</td>
<td>61</td>
<td>61</td>
<td>0</td>
<td>76.945</td>
<td>6.8906</td>
</tr>
</tbody>
</table>

Sample Report

a. What does the data represent?
I used a bag of M&Ms to test if the color percentages given by the M&M Mars Company are accurate.

b. What is the null hypothesis?
The null hypothesis is that the company is using the correct percentages in its M&M bags.

c. How do you find the expected values?
I found the expected value by multiplying the total number of M&Ms to the expected percentage for each color.

d. How do you find the chi-squared statistic?
I subtracted the observed value minus the expected value for blue M&Ms. Next, I squared this number, then divided it by the expected number. I repeated this for all the colors and added up the values in the last column to get the chi-squared statistic. That gave me approximately 6.89.

e. Explain the number of degrees of freedom:
There are six categories (because there are 6 colors) so my degrees of freedom are 6-1 or 5.

f. What two p values is this statistic between?
6.89 is between 6.63 and 7.29, so it falls between p = .25 and p = .20.

g. What is the outcome of this test?
The p value is greater than .05 so we cannot reject the null hypothesis. It appears the percentages are correct.
Mathematics | Algebra 2, Chapter 6
DATA ANALYSIS AND STATISTICS: Supplement

SPRING

Chi-Squared Table
Lesson 4
Probability p

Table entry for p is the critical value ( 2)
with the probability p lying to its right.
TABLE F:

df*
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
40
50
60
80
100

2

(

2)

distribution critical values
TAIL PROBABILITY p

.25

.20

.15

.10

.05

.025

.02

.01

.005

.0025

.001

.0005

1.32
2.77
4.11
5.39
6.63
7.84
9.04
10.22
11.39
12.55
13.70
14.85
15.98
17.12
18.25
19.37
20.49
21.60
22.72
23.83
24.93
26.04
27.14
28.24
29.34
30.43
31.53
32.62
33.71
34.80
45.62
56.33
66.98
88.13
109.1

1.64
3.22
4.64
5.99
7.29
8.56
9.80
11.03
12.24
13.44
14.63
15.81
16.98
18.15
19.31
20.47
21.61
22.76
23.90
25.04
26.17
27.30
28.43
29.55
30.68
31.79
32.91
34.03
35.14
36.25
47.27
58.16
68.97
90.41
111.7

2.07
3.79
5.32
6.74
8.12
9.45
10.75
12.03
13.29
14.53
15.77
16.99
18.20
19.41
20.60
21.79
22.98
24.16
25.33
26.50
27.66
28.82
29.98
31.13
32.28
33.43
34.57
35.71
36.85
37.99
49.24
60.35
71.34
93.11
114.7

2.71
4.61
6.25
7.78
9.24
10.64
12.02
13.36
14.68
15.99
17.28
18.55
19.81
21.06
22.31
23.54
24.77
25.99
27.20
28.41
29.62
30.81
32.01
33.20
34.38
35.56
36.74
37.92
39.09
40.26
51.81
63.17
74.40
96.58
118.50

3.84
5.99
7.81
9.49
11.07
12.59
14.07
15.51
16.92
18.31
19.68
21.03
22.36
23.68
25.00
26.30
27.59
28.87
30.14
31.41
32.67
33.92
35.17
36.42
37.65
38.89
40.11
41.34
42.56
43.77
55.75
67.50
79.08
101.9
124.3

5.02
7.38
9.35
11.14
12.83
14.45
16.01
17.53
19.02
20.48
21.92
23.34
24.74
26.12
27.49
28.85
30.19
31.53
32.85
34.17
35.48
36.78
38.08
39.36
40.65
41.92
43.19
44.46
45.72
46.98
59.34
71.42
83.30
106.6
129.6

5.41
7.82
9.84
11.67
13.39
15.03
16.62
18.17
19.68
21.16
22.62
24.05
25.47
26.87
28.26
29.63
31.00
32.35
33.69
35.02
36.34
37.66
38.97
40.27
41.57
42.86
44.13
45.42
46.69
47.96
60.44
72.61
84.58
108.1
131.1

6.63
9.21
11.34
13.28
15.09
16.81
18.48
20.09
21.67
23.21
24.72
26.22
27.69
29.14
30.58
32.00
33.41
34.81
36.19
37.57
38.93
40.29
41.64
42.98
44.31
45.64
46.96
48.28
49.59
50.89
63.69
76.15
88.38
112.3
135.8

7.88
10.60
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48.29
49.64
50.99
52.34
53.67
66.77
79.49
91.95
116.3
140.2

9.14
11.98
14.32
16.42
18.39
20.25
22.04
23.77
25.46
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28.73
30.32
31.88
33.43
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37.95
39.42
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43.78
45.20
46.62
48.03
49.44
50.83
52.22
53.59
54.97
56.33
69.70
82.66
95.34
120.1
144.3

10.83
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18.47
20.51
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43.82
45.31
46.80
48.27
49.73
51.18
52.62
54.05
55.48
56.89
58.30
59.70
73.40
86.66
99.61
124.8
149.4

12.12
15.20
17.73
20.00
22.11
24.10
24.10
27.87
29.67
31.42
33.14
34.82
36.48
38.11
39.72
41.31
42.88
44.43
45.97
47.50
49.01
50.51
52.00
53.48
54.95
56.41
57.86
59.30
60.73
62.16
76.09
89.56
102.7
128.3
153.2

* Degrees of freedom

Chi-Squared Table adapted from: www.utstat.toronto.edu/~olgac/stab22_Winter_2014/datasets/chi-sq-table.pdf

Massachusetts DYS Education Initiative—Mathematics—2017 Edition | Chapter 6, Section 14

6.14.8


OVERVIEW: The source for this activity comes from the following website.

SEE: Engage NY
www.engageny.org/resource/algebra-ii-module-4-topic-d-lesson-28/file/112046

In this lesson, you will be conducting all phases of an experiment: collecting data, creating a randomization distribution based on these data, and determining if there is a significant difference in treatment effects.

The following experiments are in homage to George E. P. Box, a famous statistician who worked extensively in the areas of quality control, design of experiments, and other topics. He earned the honor of Fellow of the Royal Society during his career and is a former president of the American Statistical Association.

The experiments will investigate whether modifications in certain dimensions of a paper helicopter will affect its flight time.

PART 1: Build the Helicopters
In preparation for your data collection, you will need to construct 20 paper helicopters following the blueprint on p. 6 of this Activity Packet.

For consistency, use the same type of paper for each helicopter. For greater stability, you may want to use a piece of tape to secure the two folded body panels to the body of the helicopter. By design, there will be some overlap from this folding in some helicopters.

You will carry out an experiment to investigate the effect of wing length on flight time.

1. Construct 20 helicopters with wing length of 4 inches and body length of 3 inches. Label 10 of each of these helicopters with the word “long.”

2. Take the other 10 helicopters, and cut 1 inch off each of the wings so that you have 10 helicopters with 3-inch wings. Label each of these helicopters with the word “short.”

3. How do you think wing length will affect flight time? Explain your answer.
PART 2: Data Collection

Once you have built the 20 helicopters, each of them will be flown by dropping the helicopter from a fixed distance above the ground (preferably 12 feet or higher—record this height for use when presenting your findings later). For consistency, drop all helicopters from the same height each time, and try to perform this exercise in a space where possible confounding factors such as wind gusts and drafts from heating and air conditioning are eliminated.

4. Place the 20 helicopters in a bag, shake the bag, and randomly pull out one helicopter. Drop the helicopter from the starting height and, using a stopwatch, record the amount of time it takes until the helicopter reaches the ground. Write down this flight time in the appropriate column in the table below. Repeat for the remaining 19 helicopters. Some helicopters might fly more smoothly than others; you may want to record relevant comments in your report.

<table>
<thead>
<tr>
<th>Trial</th>
<th>Group A: Long Wings</th>
<th>Group B: Short Wings</th>
<th>Notes and Observations</th>
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</thead>
<tbody>
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</table>

5. Why might it be important to randomize (impartially select) the order in which the helicopters were dropped? (This is different from the randomization you will perform later when you are allocating observations to groups to develop the randomization distribution.)
PART 3: Analyzing Your Data

Now that you have your data, you can analyze the effect of wing length. In this experiment, you are investigating whether wing length makes a difference in flight time. You are comparing the helicopters with long wings (wing length of 4 inches, Group A) to the helicopters with short wings (wing length of 3 inches, Group B). Since you dropped the helicopters from the same height in the same location, using the same type of paper, the only difference in the two groups should be the different wing lengths.

How to Create a Randomization Distribution

You will use the computer to carry out a randomization test to answer these questions. Ask your teacher to set you up with the Data Analysis tool on your computer. When the tool opens, you will see “Data Sheet 1”—just like a spreadsheet. Use Column 1 to enter the data from Group A helicopters and then Column 2 to enter the data from Group B. Once the data is entered:

Choose File ➔ Save to save the data.

Now you are ready to analyze the data. First, find out which group of helicopters had the longer flight times.

Choose Statistics ➔ Descriptive to see the mean flight time for each group. Record the statistics here:

Mean of Group A: _____________
Mean of Group B: _____________

Find the difference between the means: _____________

So, is it a significant difference? Does it really matter how long the helicopter’s wings are? Or did we happen to see longer flight times in one of the groups, just by chance? You will use a randomization distribution to answer these questions.

Choose Statistics ➔ Randomization Distribution and then hit OK when it asks about the columns.

A new window will open and now you are ready to build your distribution. We want to know if the difference you saw could have happened just by chance. So we have to look at the kinds of things that can happen due to chance. The computer will be taking the red numbers and the blue numbers and shuffling them together. Then, from the mixed pile of numbers, the computer will randomly choose 10 of them to go in Column A. The other 10 go in Column B.

Hit the Green Arrow Button to see this in action.

Your two random samples will have new means and the computer will find the difference between the means. Because the samples were picked randomly, this new result is an example of a difference in means that can happen just by chance. You will see that difference in means recorded as a gray bar on the histogram.

Of course, lots of things could happen depending on how the red and blue numbers get shuffled. So you want to repeat that shuffling experiment many times. Hit the double blue arrow button a few times to make the shuffling experiment happen repeatedly.
You will see the histogram start to fill in, showing the results of all the different things that can happen. Some results happen pretty often, while other results are more unlikely. Let it run 5,000 times.

**Hit the Red Stop Button.**

**Choose Options** and then **Check the Boxes** for ✓ Relative Frequencies and ✓ Label Bars.

Now you can see the percentages in each bar of the histogram. This tells you how often a certain result happened. It might be 16.3% of the time or 2.1% of the time—something like that.

**How to Read a Randomization Distribution**

On the histogram, you will see a vertical red line showing the original difference in means between the Group A helicopters and the Group B helicopters. This red line is going to help you decide if your difference is significant or not. Where is your red line compared to the histogram? Is it pretty near the middle of the histogram, or is it way out on one of the tails? Use the bars of the histogram to add up all the probabilities of getting something more extreme than the red line. This tells you the probability of getting that result just by chance. How likely is it?

**PART 4: Stating Your Final Conclusions**

Remember that we started this experiment to answer certain questions:

Does a 1-inch addition in wing length appear to result in a change in average flight time? If so, which helicopters tend to have longer flight times—the ones with longer wing length or the ones with shorter wing length?

You should now write a final conclusion that clearly answers the questions about the helicopters. Back up your conclusion by referring to your data and the analysis that you did with the randomization distribution.

**EXTENSION**

One other variable that can be adjusted in the paper helicopters is body width. See the blueprints for details.

1. Construct 10 helicopters using the blueprint from the lesson. Label each helicopter with the word “narrow.”
2. Develop a blueprint for a helicopter that is identical to the blueprint used in class except for the fact that the body width will now be 1.75 inches.
3. Use the blueprint to construct 10 of these new helicopters, and label each of these helicopters with the word “wide.”
4. Place the 20 helicopters in a bag, shake the bag, and randomly pull out one helicopter. Drop the helicopter from the starting height and, using a stopwatch, record the amount of time it takes until the helicopter reaches the ground. Write down this flight time in the appropriate column in the table on Extension Worksheet. Repeat for the remaining 19 helicopters.
5. Finally, answer the questions and complete the tasks at the bottom of the Extension Worksheet.
EXENSION WORKSHEET: Record flight times in this table for your 20 helicopters as instructed in item 4 of the Extension activities, then complete the questions and tasks below.

<table>
<thead>
<tr>
<th>Trial</th>
<th>Group A: Narrow Body</th>
<th>Group B: Wide Body</th>
<th>Notes and Observations</th>
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</thead>
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<td>1</td>
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</table>

**Questions:**
Does a 0.5-inch addition in body width appear to result in a change in average flight time?
If so, do helicopters with wider body width (Group C) or narrower body width (Group D) tend to have longer flight times on average?

Carry out a complete randomization test to answer these questions. Show all 5 steps, and use the Data Analysis Tool to assist both in creating the distribution and with your computations.
Be sure to write a final conclusion that clearly answers the questions in context.
DIRECTIONS:
To make helicopter, cut on SOLID lines and fold on DASHED lines. Overall size is 8 x 4 inches.

For greater stability, use a piece of tape to secure the two folded back panels to the body of the helicopter.
DYS Pedagogical Practices Links ................................................................. A.1
100 Questions that Promote Mathematical Discourse ................................. A.3
DYS Pedagogical Practices Links

Comprehensive Educational Partnership (CEP)

Massachusetts Department of Youth Services
www.mass.gov/dys

Commonwealth Corporation
commcorp.org

Collaborative for Educational Services
www.ccollaborative.org

Massachusetts Department of Elementary and Secondary Education (ESE)

Massachusetts Department of Elementary and Secondary Education
www.doe.mass.edu

Massachusetts Curriculum Frameworks (all current)
www.doe.mass.edu/frameworks/

Next-Generation MCAS
www.doe.mass.edu/mcas/nextgen/

Common Core State Standards (CCSS)

Common Core State Standards
www.corestandards.org/
www.corestandards.org/other-resources/key-shifts-in-mathematics/

Common Core Shifts
www.engageny.org/resource/common-core-shifts

ISTE Standards
www.iste.org/standards

Career and College Readiness (CCR)

Achieve.org
www.achieve.org/college-and-career-readiness
Pedagogy

Access for All

Bloom’s Taxonomy

Differentiated Instruction
www.ascd.org/professional-development/differentiated-classroom-2nd-edition.aspx?gclid=CNKq-8jUn8ECFFlMm7AodVVgABQ

Empower Your Future Curricula (EYF)
commcorp.org/resources/?cat=resources_curriculum

Essential Questions
www.authenticeducation.org/ae_bigideas/article.lasso?artid=53

New Visions for Public Schools
https://curriculum.newvisions.org/math

Positive Youth Development in DYS Educational Programs (PYD/CRP)
www.commcorp.org/resources/detail.cfm?ID=705

Understanding by Design (UbD)

Understanding by Design Resources
www.ascd.org/research-a-topic/understanding-by-design-resources.aspx

What is Understanding by Design?
www.authenticeducation.org/ubd/ubd.lasso

Universal Design for Learning (UDL)

About Universal Design for Learning
www.cast.org/udl

Universal Design for Learning Chart
www.udlcenter.org/aboutudl/udlguidelines/udlguidelines_graphicorganizer

Universal Design for Learning Wheel

Figure A: (two parts: A+B)
www.menacommoncore.com/2013/Reem_Labib_MENA_Common_Core_ULD%20DIY%20Template%20Wheel.jpg

Figure B:
www.menacommoncore.com/2013/Reem_Labib_MENA_Commo_Core_ULD%20DIY%20Template%20Wheel-1.jpg
Pedagogy (continued)

WIDA Consortium

WIDA Online Download Library
www.wida.us/downloadLibrary.aspx

WIDA Booklet with Performance Definitions, 9-12
www.wida.us/get.aspx?id=22

100 Questions that Promote Mathematical Discourse

The following resource is reprinted on the next four pages with permission from Ready Marketing/ Curriculum Associates. A PDF download is also available online.

100 Questions that Promote Mathematical Discourse

Google Drive Math Guide Resources | Appendix
http://bit.ly/2qp5RVb (DYS/SEIS educators only)

Massachusetts Department of Youth Services Instructional Guides, including this Mathematics Guide, are available as PDF downloads.

See: Collaborative for Educational Services
https://www.collaborative.org/programs/dys/dys-instructional-guides

Please note:
There are a number of supporting documents that are available to our teachers via a Google Drive Math Guide resource for DYS/SEIS educators. We are not able to provide a public link at this time.
100 questions that promote Mathematical Discourse

Help students work together to make sense of mathematics

1. What strategy did you use?
2. Do you agree?
3. Do you disagree?
4. Would you ask the rest of the class that question?
5. Could you share your method with the class?
6. What part of what he said do you understand?
7. Would someone like to share ___?
8. Can you convince the rest of us that your answer makes sense?
9. What do others think about what [student] said?
10. Can someone retell or restate [student]’s explanation?
11. Did you work together? In what way?
12. Would anyone like to add to what was said?
13. Have you discussed this with your group? With others?
14. Did anyone get a different answer?
15. Where would you go for help?
16. Did everybody get a fair chance to talk, use the manipulatives, or be the recorder?
17. How could you help another student without telling them the answer?
18. How would you explain ___ to someone who missed class today?

Help students rely more on themselves to determine whether something is mathematically correct

19. Is this a reasonable answer?
20. Does that make sense?
21. Why do you think that? Why is that true?
22. Can you draw a picture or make a model to show that?
23. How did you reach that conclusion?
24. Does anyone want to revise his or her answer?
25. How were you sure your answer was right?

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Help students learn to reason mathematically

26 How did you begin to think about this problem?
27 What is another way you could solve this problem?
28 How could you prove ______?
29 Can you explain how your answer is different from or the same as [student]'s answer?
30 Let's break the problem into parts. What would the parts be?
31 Can you explain this part more specifically?
32 Does that always work?
33 Can you think of a case where that wouldn't work?
34 How did you organize your information? Your thinking?

Help students evaluate their own processes and engage in productive peer interaction

35 What do you need to do next?
36 What have you accomplished?
37 What are your strengths and weaknesses?
38 Was your group participation appropriate and helpful?
39 What is this problem about? What can you tell me about it?
40 Do you need to define or set limits for the problem?
41 How would you interpret that?
42 Could you reword that in simpler terms?
43 Is there something that can be eliminated or that is missing?
44 Could you explain what the problem is asking?
45 What assumptions do you have to make?
46 What do you know about this part?
47 Which words were most important? Why?

Help students with problem comprehension

32 33 34
35 36 37 38
39 40 41 42
43 44 45 46
47

Reprinted with permission from Ready Marketing/Curriculum Associates
Help students learn to **conjecture, invent, and solve** problems

48. **What would happen if** ___?
49. Do you see a **pattern**?
50. What are some **possibilities** here?
51. Where could you find the **information** you need?
52. How would you **check your steps** or your answer?
53. What **did not work**?
54. How is your solution method the **same as or different from** [student]’s method?
55. Other than retracing your steps, **how can you determine** if your answers are appropriate?
56. How did you **organize** the information? Do you have a **record**?
57. How could you solve this using **tables, lists, pictures, diagrams**, etc.?
58. What have you tried? What **steps** did you take?
59. How would it look if you used this **model** or these **materials**?
60. How would you draw a **diagram or make a sketch** to solve the problem?
61. Is there another **possible answer**? If so, explain.
62. Is there another **way to solve** the problem?
63. Is there another **model** you could use to solve the problem?
64. Is there anything you’ve **overlooked**?
65. **How did you think** about the problem?
66. What was your **estimate or prediction**?
67. How **confident** are you in your answer?
68. **What else** would you like to know?
69. What do you think comes **next**?
70. Is the solution **reasonable**, considering the context?
71. Did you have a **system**? Explain it.
72. Did you have a **strategy**? Explain it.
73. Did you have a **design**? Explain it.
Help students learn to connect mathematics, its ideas, and its application

74 What is the relationship between ____ and ____?
75 Have we ever solved a problem like this before?
76 What uses of mathematics did you find in the newspaper last night?
77 What is the same?
78 What is different?
79 Did you use skills or build on concepts that were not necessarily mathematical?
80 Which skills or concepts did you use?
81 What ideas have we explored before that were useful in solving this problem?
82 Is there a pattern?
83 Where else would this strategy be useful?
84 How does this relate to ____?
85 Is there a general rule?
86 Is there a real-life situation where this could be used?
87 How would your method work with other problems?
88 What other problem does this seem to lead to?
89 Have you tried making a guess?
90 What else have you tried?
91 Would another method work as well or better?
92 Is there another way to draw, explain, or say that?
93 Give me another related problem. Is there an easier problem?
94 How would you explain what you know right now?
95 What was one thing you learned (or two, or more)?
96 Did you notice any patterns? If so, describe them.
97 What mathematics topics were used in this investigation?
98 What were the mathematical ideas in this problem?
99 What is mathematically different about these two situations?
100 What are the variables in this problem? What stays constant?